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Navier–Stokes problems with random coefficients by the Weighted Least Squares Technique Stochastic Finite Volume Method

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ABSTRACT

The main aim of this article is numerical solution to the Navier–Stokes equations for incompressible, non-turbulent and subsonic fluid flows with Gaussian physical random parameters. It is done with the use of the specially adopted Finite Volume Method extended towards probabilistic analysis by the generalized stochastic perturbation technique. The key feature of this approach is the weighted version of the Least Squares Method implemented symbolically in the system MAPLE to recover nodal polynomial response functions of the velocities, pressures and temperatures versus chosen input random variable(s). Such an implementation of the Stochastic Finite Volume Method is applied to model 3D flow problem in the statistically homogeneous fluid with uncertainty in its viscosity and, separately, coefficient of the heat conduction. Probabilistic central moments of up to the fourth order and the additional characteristics are determined and visualized for the cavity lid driven flow owing to the specially adopted graphical environment FEPlot. Further numerical extension of this technique is seen in an application of the Taylor–Newton–Gauss approximation technique, where polynomial approximation may be replaced with the exponential or hyperbolic ones.

1. Introduction

Computational solution of the fully coupled Navier–Stokes equations is still really challenging problem, especially when defined in terms of random coefficients. We prefer the generalized stochastic perturbation technique as it allows for a determination of third and fourth central moments as well as such coefficients like skewness and/or kurtosis. Instead of a time consuming implementation of the Direct Differentiation Method (DDM), the Response Function Method (RFM) is preferred, so that instead of up to the nth order coupled Navier–Stokes equations we solve for some polynomial approximations of the state functions relating the PVT solution with the input random variable(s). This approximation is proposed here in a local sense – the response functions for velocities, pressures and temperatures may be different in each discrete point of the computational domain. This idea is connected here with the classical deterministic formulation of the Finite Volume Method (FVM) [2–4]. A very useful property of the FVM is that the conservation principles, which are the basis for the mathematical modeling of continuum mechanical problems are also

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Notation

Roman symbols

c	specific heat
d_j	direction vector
$\tilde{f}_i$	body forces per unit volume
g	gravitational acceleration
k	thermal conductivity
n_s	number of the finite volume outer faces
p	fluid pressure
$p_b(x)$	probability density function
q_i	the heat flux
t	time parameter
v_i	velocity vector
A_j	an area the face j of the given finite volume
$E[b]$	expected value of random variable b
$D^P_{\beta m}$	matrix of unknown polynomial coefficients for pressure response
$D^T_{\beta m}$	matrix of unknown polynomial coefficients for temperature response
$D^U_{\beta m}$	matrix of unknown polynomial coefficients for velocity response
$K_l^{P(\alpha)}, \overline{K}_{lj}^{P(\alpha)}$	system matrices for the pressures corresponding to the lth finite volume center and the center of its jth outer face, αth RFM test
$K_l^{T(\alpha)}, \overline{K}_{lj}^{T(\alpha)}$	system matrices for the temperatures corresponding to the lth finite volume center and the center of its jth outer face, αth RFM test
$K_l^{U(\alpha)}, \overline{K}_{lj}^{U(\alpha)}$	system matrices for the velocities corresponding to the lth finite volume center and the center of its jth outer face, αth RFM test
M	total number of deterministic experiments necessary for the response function recovery
N	total number of degrees of freedom in the system
$\mathbf{P}$	vector of discrete pressures
$P_l^{(\alpha)}(t)$	pressure in the center of finite volume l at time t, αth RFM test
$\overline{P}_{lj}^{(\alpha)}(t)$	pressure face flux (finite volume l, its j outer plane, at time t, αth RFM test)
$Q_l^{P(\alpha)}, Q_l^{U(\alpha)}, Q_l^{T(\alpha)}$	the R.H.S. vectors for the pressures, velocities and temperatures at the lth finite volume and αth RFM test
S_j	normal vector
$\mathbf{T}$	discrete temperatures vector
$T_l^{(\alpha)}(t)$	temperature of the center of finite volume l at time t, αth RFM test
$\overline{T}_{lj}^{(\alpha)}(t)$	temperature face flux (finite volume l, its j outer plane, at time t, αth RFM test)
$\mathbf{U}$	the vector of discrete velocities
$U_l^{(\alpha)}(t)$	velocity of the center of finite volume l at time t, αth RFM test
$\overline{U}_{lj}^{(\alpha)}(t)$	velocity face flux (finite volume l, its j outer plane, at time t, αth RFM test)
V_l	total volume of the lth sub-volume
$\mathrm{Var}(b)$	variance of random variable b

Greek symbols

α, β	the local index symbol
$\alpha(b)$	the coefficient of variation of random variable b
$\beta(b)$	skewness coefficient of random variable b
$\kappa(b)$	kurtosis of random variable b
δ_{ij}	Kronecker delta
ε_{ij}	strain tensor
ε	perturbation parameter
ρ	fluid density
μ	fluid viscosity
$\mu_p(b)$	pth central moment of the variable b
σ_{ij}	stress tensor
χ	interpolation coefficient
θ	temperature
$\phi_l^{(\alpha)}$	viscous dissipation for the lth finite volume and αth RFM numerical test
φ_β	shape functions
$(\cdots)_{,i}$	partial derivative symbol
Δt	time increment

fulfilled for the discrete equations [8]. A starting point for the FVM is a decomposition of the problem domain into both regular and irregular sub-volumes, where each such a sub-volume is represented by its midpoint only. This is the main difference to the Finite Element Method (FEM) [5,8], where the equilibrium equations are formed and solved in the nodal points of the mesh only, which are located in the corners (and midpoints for higher order approximations) of each finite element.

Computational analysis is provided in a hybrid way here – the FVM freeware code OpenFVM is engaged to solve all N–S problems necessary to build up the response functions. The internal symbolic Least Squares Method of the system MAPLE accompanied with the perturbation-based formulas implemented in this program leads to the final statistical moments of the fluid state. We recommend the weighted version of the LSM, where each discretization point to define variability of the input random quantity has some associated weight showing its contribution to the final expected value. Numerical visualization is carried out in the freeware FEPlot used before for the FEM and FDM output files and procedures. Computational illustration deals with incompressible fluid flow in a cubic domain and this flow occurs with two Gaussian input random variables – heat conductivity coefficient and, separately, fluid viscosity. We compute twice up to fourth order probabilistic characteristics of the PVT solution to validate an importance of both physical parameters. Although these input parameters are state-independent, further extension of the proposed SFVM toward numerical modeling of nonlinear, i.e. temperature-dependent systems will be also possible.

2. Governing equations

2.1. Navier–Stokes equations

The system of basic equilibrium equations to be extended toward stochastic analysis and to be solved numerically can be written with boundary conditions as follows [2,6,7]: