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The combined asymptotic–tolerance model of heat conduction in a skeletal micro-heterogeneous hollow cylinder

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ABSTRACT

In this paper a new model describing heat transfer in cylindrical composite conductor with functionally graded macro-structure and deterministic tolerance-periodic micro-structure is presented. The formulation of this model will be based on the known Fourier's theory of heat conduction and additional hypothesis of the tolerance averaging approach (Woźniak et al., 2008, 2010). The comparison of the obtained new model with tolerance model and finite element method will be shown. The general results will be applied to analysis of some special problems.

1. Introduction

The object of consideration is a rigid composite conductor with functionally graded macro-structure. This consideration deals with two-phase hollow cylinder made of homogeneous matrix and homogeneous walls of constant width which are uniformly distributed along angular direction (Fig. 1(a)). It is assumed that a cross section of the cylindrical conductor perpendicular to the walls mid-surfaces represent a certain micro-heterogeneous plane structure which is, for a fixed radius, periodic along ξ^1-coordinate but have slowly varying apparent properties in the radial direction (Fig. 1(b)). In considered conductor the number of walls n is very large ($n^{-1} \ll 1$). Hence the period λ of heterogeneity is assumed to be sufficiently small when compared to the measure of the domain of argument ξ^1 (Fig. 1).

Since the direct description of the conductor under consideration leads to equation with highly oscillating, tolerance periodic and discontinuous coefficients, it cannot be a proper tool to engineering problems analysis and numerical calculations for such conductors. The obtained averaged model equations with continuous, smooth and slowly varying functional coefficients replace micro-heterogeneous conductor by conductor with averaged properties. Conductor of this kind is treated as made of *functionally graded material*, cf. [14].

Functionally graded structures often have not deterministic micro-structures, but they are composition of constituents generating continuous and smooth gradation of apparent properties. The analysis of the heat transfer in a hollow cylinder made of functionally graded materials can be found for example in [9,13,15], where material properties are expressed as power or exponential functions of the radial coordinate. The hollow cylinder presented in [2] has a heterogeneous micro-structure and it is divided into many subcylinders (layers) across the thickness. Ootao [8] studied transient thermoelastic problem for multilayered hollow sphere, where the thermal properties of each layer were expressed as power functions of the radial coordinate. In contrast to the above papers, the analysed conductor has a deterministic micro-structure.

The formulation of the averaged mathematical model to the skeletal conductor will be based on the tolerance averaging approach. This modelling technique describes the effect of the micro-structure size on the overall response of a composite structure. The general modelling procedures of this technique are given in books by Woźniak et al. [16,18].

The application of this technique for modelling and analysis of heat conduction in periodic composites were presented in a series of papers, e.g. [5,4,7,11,17]. In books edited by Woźniak et al. [16,18] the extended list of references on this subject can be found. The tolerance averaging technique was also adopted and applied to formulate mathematical models and to analyse heat conduction problems for functionally graded stratified solids, e.g. [3,6,10,12,19].

The main aim of presented paper is to propose a new combined asymptotic–tolerance model of the functionally graded heat conductor. This contribution is a certain continuation of the paper by Ostrowski and Michalak [10] but in its contrast takes into account the effect of the radial walls for the heat flux in the circumferential

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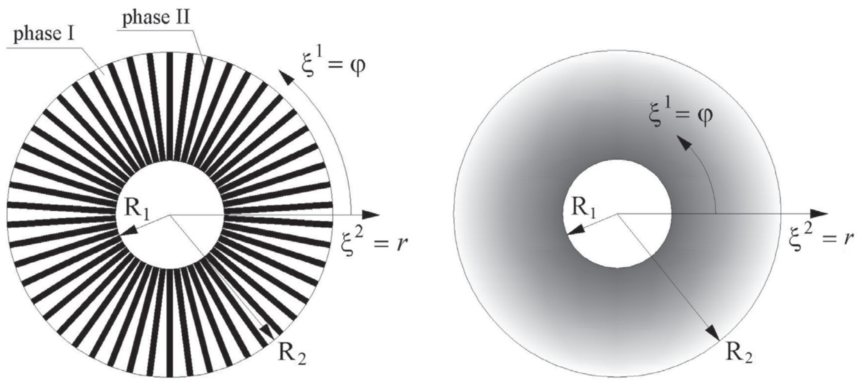

(a) micro-scale (b) macro-scale

Fig. 1. Composite structure.

and radial direction. Application of the averaged combined model equations to special problems of heat conduction are also shown.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^3$ be a region in Euclidean space occupied by the conductor under consideration in which we introduce a cylindrical coordinate system $O\xi^1\xi^2\xi^3$. Denote

$$\Pi := (0, 2\pi) \subset \mathbb{R}, \quad \Xi := (R_1, R_2) \times (-H, H) \subset \mathbb{R}^2, \tag{1}$$

as bounded and regular regions, where $H > 0$. If $H = \infty$ then $\Omega = \Pi \times \Xi$ describes the region occupied by an infinite hollow cylinder in $\mathbb{R}^3$. Points from Π will be denoted by $\xi \equiv \xi^1$ or $\eta \equiv \eta^1$ and from Ω by $\boldsymbol{\xi} = (\xi^1, \xi^2, \xi^3)$. Points from Ξ will be denoted by $\zeta = (\xi^2, \xi^3)$.

Through this paper gradient operators are $\partial = (\partial_1, 0, 0)$, $\overline{\nabla} = (0, \partial_2, \partial_3)$ and $\nabla = (\partial_1, \partial_2, \partial_3)$, where $\partial_i = \partial/\partial\xi^i, i = 1, 2, 3$, stand for covariant derivatives in cylindrical coordinate system. Dots over name of the function stand for the time derivatives. Indexes with letters i,j run over 1, 2, 3 and indexes with Greek letters α, β run over 2, 3. Summation convention holds for all aforementioned indexes.

The basic law describing heat conduction in a medium is well known Fourier's theory

$$\nabla \cdot (\mathbf{K} \cdot \nabla\Theta) - c\rho \cdot \dot{\Theta} = \dot{Q}, \tag{2}$$

where $\mathbf{K} = [k^{ij}], k^{ij} : \Omega \to \mathbb{R}$ stands for the second order conductivity tensor, $c : \Omega \to \mathbb{R}$ is specific heat and $\rho : \Omega \to \mathbb{R}$ is density of the material. Intensity of internal heat sources is denoted by $\dot{Q} : \Omega \times [t_0, t_1] \to \mathbb{R}$. Function of the temperature $\Theta : \Omega \times [t_0, t_1] \to \mathbb{R}$ has to satisfy Eq. (2) for every $\boldsymbol{\xi} \in \Omega$ and $t \in [t_0, t_1]$. This direct description leads to the equation with highly oscillating coefficients, which is too complicated to be used in the engineering analysis of the heat conduction problems and numerical calculations.

We assume in further considerations that conductor has constant material properties and invariable structure in ξ^3 direction. Therefore our problem will be restricted only to the $O\xi^1\xi^2$ plane and ξ^3 axis will be simply omitted in every aspect of this contribution.

3. Modelling concepts

Let $n \in \mathbb{N}$ such that $n^{-1} \ll 1$. Assume that considered composite consists of n mutually not intersecting, uniformly and circumferential distributed homogeneous walls of constant width $a > 0$ in ξ^2 direction (Fig. 2). The space between the walls in Ω is filled by the second homogeneous material, called matrix. In other words, regions Ω_w, Ω_m occupied by the walls and matrix, respectively, can be presented as

$$\Omega_w := \bigcup_{m=0}^{n-1} \left\{ \boldsymbol{\xi} \in \Omega : \quad \left|\xi^1 - m\lambda\right| < \frac{a}{2\xi^2}, \ \xi^2 \in (R_1, R_2) \right\}, \qquad \Omega_m := \Omega - \overline{\Omega}_w, \tag{3}$$

where $\lambda = 2\pi/n$. It is easy to see that the walls not-intersecting condition to be true, the inequality $a < \lambda R_1$ must hold. Moreover, for a fixed ξ^2-coordinate we deal here with periodic structure. Since n is large enough, the micro-structure parameter λ is sufficiently small compared to the smallest characteristic length dimension L_Π of Π ($\lambda \ll L_\Pi$).

Directly from representation (3) we can distinguish an arbitrary unit cell $\Delta(\xi) := \xi + \Delta$ with the centre at $\xi \in \overline{\Omega}$, where

$$\Delta := \left\{ \boldsymbol{\xi} \in \Omega : \quad \left|\xi^1\right| < \frac{\lambda}{2} \right\} = \left(-\frac{\lambda}{2}, \frac{\lambda}{2} \right) \times \Xi, \tag{4}$$

in which we introduce a local coordinate system $O\eta^1\eta^2\eta^3$. Hence, for $\xi \in \overline{\Omega}$ we denote

$$\Omega_\xi := \Omega \cap \bigcup_{\boldsymbol{\eta} \in \Delta(\xi)} \Delta(\boldsymbol{\eta}) \tag{5}$$

as a cluster of 2 cells having common sides.

In order to derive averaged model equations we applied tolerance averaging approach. This technique is based on the concept of tolerance and in-discernibility relation. The general modelling procedures and basic definitions and theorems of this technique are given in books [16,18]. We mention here some basic concepts of this technique, as a locally periodic function, an averaging

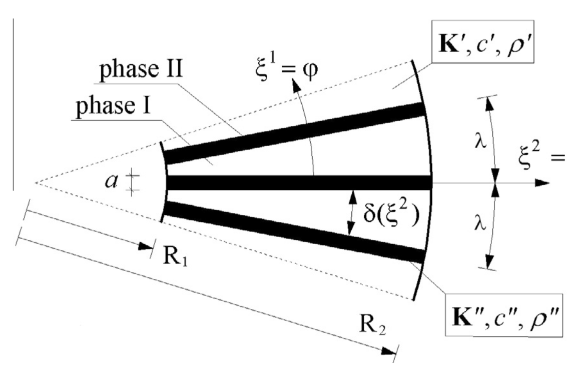

Fig. 2. A unit cell micro-structure.