Composite Structures 134 (2015) 593–601

Contents lists available at ScienceDirect

Composite Structures

journal homepage: www.elsevier.com/locate/compstruct

A simple and accurate generalized shear deformation theory for beams

J.L. Mantari [a,*], J. Yarasca [b]

[a] Faculty of Mechanical Engineering, University of Engineering and Technology, Jr. Medrano Silva 165, Barranco, Lima, Peru
[b] Faculty of Mechanical Engineering, National University of Engineering, Av. Túpac Amaru 210, Rimac, Lima, Peru

ARTICLE INFO

Article history:
Available online 28 August 2015

Keywords:
Beams
Elasticity
Bending
Analytical modeling

ABSTRACT

This paper presents a static analysis of functionally graded (FG) single and sandwich beams by using a simple and efficient 4-unknown quasi-3D hybrid type theory, which includes both shear deformation and thickness stretching effects. The governing equations and boundary conditions are derived by employing the principle of virtual works. Navier-type closed-form solution is obtained for several beams. New hybrid type shear strain shape functions for the inplane and transverse displacement were introduced in general manner to model the displacement field of beams. Numerical results of the present compact quasi-3D theory are compared with other quasi-3D higher order shear deformation theories (HSDTs).

1. Introduction

Functionally graded materials (FGMs) are a type of heterogeneous composite material in which the properties change gradually over one or more directions. FGMs made possible to avoid abrupt changes in the stress and displacement distributions. Currently, FGMs are alternative materials widely used in aerospace, nuclear reactor, energy sources, biomechanical, optical, civil, automotive, electronic, chemical, mechanical, and shipbuilding industries.

FGMs were proposed by Bever and Duwez [1], and after them several researchers have provided results on functionally graded plates [2–10], sandwich plates [11,12], shells [13,14] and beams [15–17]; this short list gives an idea of some contribution in the field. Carrera et al. [18] investigated the influence of the stretching effect on the static responses of functionally graded (FG) plates and shells, which is especially significant for thick FG plates. Consequently, thickness stretching effects is also necessary to include in beam formulations for the precise mechanical prediction of stresses.

As far as the authors are aware, there is limited work available for bending analysis of FG sandwich beams. Vo et al. [19] develop a quasi-3D polynomial theory with 4 unknowns to investigate the static behavior and the effect of normal strain in FG sandwich beams for various power-law index, skin-core-skin thickness ratios and boundary conditions. In this context, the influence of non-polynomial or hybrid type shear strain shape functions were not explored to study FG beams along with C^1 HSDTs. However, it is remarkable to mention the work by Filippi et al. [20] based on Giunta et al. [15–17] beam formulation (1D Carrera's unified formulation), where trigonometric, polynomial, exponential and miscellaneous expansions are used and evaluated for various structural problems. This paper attempts to cover this gap.

In this paper, a 4-unknown hybrid type quasi-3D theory with both shear deformation and thickness stretching effects for the bending analysis of FG beams is presented. Many quasi-3D hybrid type (polynomial, non-polynomial, and hybrid) HSDTs, including the thickness expansion can be derived by using the present generalized theory. The theory complies with the tangential stress-free boundary conditions on the beam boundary surface, and thus a shear correction factor is not required. The beam governing equations and its boundary conditions are derived by employing the principle of virtual works. Navier-type analytical solution is obtained for sandwich beams subjected to transverse load for simply supported boundary conditions. The results are compared with other quasi-3D HSDT and further referential results for the displacement and stresses of FG sandwich beams are obtained.

2. Analytical modeling of FG beams

An FG beam of length a, width b and a total thickness h made of a mixture of metal and ceramic materials are considered in the present analysis. The elastic material properties vary through the thickness and the power-law distribution [19]:

$$E(z) = (E_c - E_m)V_c(z) + E_m, \tag{1}$$

* Corresponding author. Tel.: +51 1 3540070; cell: +51 96224551.
E-mail address: jmantari@utec.edu.pe (J.L. Mantari).

http://dx.doi.org/10.1016/j.compstruct.2015.08.073
0263-8223/

where subscripts m and c represent the metallic and ceramic constituents, V_c is the volume fraction of the ceramic phase of the beam. For comparison reasons, three types of FG beams are considered, see Fig. 1.

2.1. Type A FG beams

The beam is composed of a FG material (Fig. 1a) with V_c given by:

$$V_c(z) = \left(\frac{2z+h}{2h}\right)^p. \tag{2}$$

2.2. Type B sandwich beams with homogeneous skins and FG core

The bottom and top skin of sandwich beams is metal and ceramic, while, the core is composed of a FG material (Fig. 1b) with V_c given by [19]:

$$\begin{aligned} &V_c = 0 && z \in [-h/2, h_1] && \text{(bottom skin)},\\ &V_c = \left(\tfrac{z-h_1}{h_2-h_1}\right)^p && z \in [h_1, h_2] && \text{(core)},\\ &V_c = 1 && z \in [h_2, h/2] && \text{(top skin)}. \end{aligned} \tag{3}$$

2.3. Type C sandwich beams with FG skins and ceramic core

The bottom and top skin of sandwich beams is composed of a FG material, while, the core is ceramic (Fig. 1c) with V_c given by [19]:

$$\begin{aligned} &V_c = \left(\tfrac{z-h_0}{h_1-h_0}\right)^p && z \in [-h/2, h_1] && \text{(bottom skin)},\\ &V_c = 1 && z \in [h_1, h_2] && \text{(core)},\\ &V_c = \left(\tfrac{z-h_3}{h_2-h_3}\right)^p && z \in [h_2, h/2] && \text{(top skin)}. \end{aligned} \tag{4}$$

2.4. Theoretical displacement field

The displacement field satisfying the free surfaces boundary conditions of transverse shear stresses (and hence strains) vanishing at a point $(x, \pm h/2)$ on the outer (top) and inner (bottom) surfaces of the beam, is given as follows:

$$\begin{aligned} \overline{u}(x,z) &= u + z\left[y^{**}\frac{\partial w_b}{\partial x} + q^*\frac{\partial \theta}{\partial x} - \frac{\partial w_s}{\partial x}\right] + f(z)\frac{\partial w_b}{\partial x},\\ \overline{w}(x,z) &= w_b + w_s + g(z)\theta, \end{aligned} \tag{5}$$

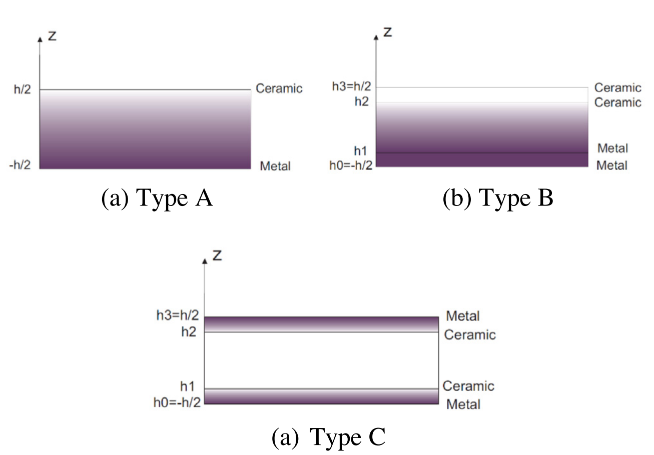

Fig. 1. Geometry and coordinate of a FG sandwich beam.

where u, w_s, w_b and θ are four unknown displacements of midplane of the beam. The constants y^{**}, y^* and q^* are obtained by considering the criteria to reduce the number of unknowns in HSDTs as in Reddy and Liu [21]. They are as a function of the shear strain shape functions, $f(z)$ and $g(z)$, i.e. $y^{**} = y^* - 1$, $y^* = -f'\left(\frac{h}{2}\right)$ and $q^* = -g\left(\frac{h}{2}\right)$.

For deriving the equations, small elastic deformations are assumed, i.e. displacements and rotations are small, and obey Hookes law. The starting point of the present generalized quasi-3D HSDT is the 3D elasticity theory [22]. The strain–displacement relations, based on this formulation, are written as follows:

$$\begin{aligned} \varepsilon_{xx} &= \varepsilon^0_{xx} + z\varepsilon^1_{xx} + f(z)\varepsilon^2_{xx},\\ \varepsilon_{zz} &= g'(z)\varepsilon^5_{zz},\\ \gamma_{xz} &= \gamma^0_{xz} + g(z)\gamma^3_{xx} + f'(z)\gamma^4_{xz}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} &\varepsilon^0_{xx} = \tfrac{\partial u}{\partial x} && \varepsilon^1_{xx} = y^{**}\tfrac{\partial^2 w_b}{\partial x^2} + q^*\tfrac{\partial^2\theta}{\partial x2} - \tfrac{\partial^2 w_s}{\partial x^2} && \varepsilon^2_{xx} = \tfrac{\partial^2 w_b}{\partial x^2},\\ &\varepsilon^5_{zz} = \theta, && &&\\ &\varepsilon^0_{xz} = y^*\tfrac{\partial w_b}{\partial x} + q^*\tfrac{\partial \theta}{\partial x} && \varepsilon^3_{xz} = \tfrac{\partial\theta}{\partial x} && \varepsilon^4_{xz} = \tfrac{\partial w_b}{\partial x}, \end{aligned} \tag{7}$$

The linear constitutive relations are given below:

$$\begin{Bmatrix} \sigma_{xx}\\ \sigma_{zz}\\ \tau_{xz} \end{Bmatrix}_{(z)} = \begin{bmatrix} Q_{11} & Q_{13} & 0\\ Q_{13} & Q_{33} & 0\\ 0 & 0 & Q_{55} \end{bmatrix}_{(z)} \begin{Bmatrix} \varepsilon_{xx}\\ \varepsilon_{zz}\\ \gamma_{xz} \end{Bmatrix}_{(z)}, \tag{8}$$

in which, $\sigma_{(z)} = \{\sigma_{xx}, \sigma_{zz}, \tau_{xz}\}^T$ and $\varepsilon_{(z)} = \{\varepsilon_{xx}, \varepsilon_{zz}, \gamma_{xz}\}^T$ are the stresses and the strain vectors with respect to the beam coordinate system. The Q_{ij} expressions are given below:

$$\begin{aligned} Q_{11(z)} &= Q_{33(z)} = \frac{E(z)}{1-\nu^2},\\ Q_{13(z)} &= \frac{E(z)\nu}{1-\nu^2},\\ Q_{55(z)} &= \frac{E(z)}{2(1+\nu)}. \end{aligned} \tag{9}$$

The elastic coefficients Q_{ij} vary through the thickness according to Eq. (1).

Considering the static version of the principle of virtual work, the following expressions can be obtained:

$$0 = \left[\int_{-h/2}^{h/2}\left\{\int_{\Omega}[\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{zz}\delta\varepsilon_{zz} + \tau_{xz}\delta\gamma_{xz}]dxdy\right\}dz\right] - \left[\int_{\Omega} q\delta\overline{w}\,dxdy\right], \tag{10}$$

$$\begin{aligned} 0 = \int_{\Omega} &(N_1\delta\varepsilon^0_{xx} + M_1\delta\varepsilon^1_{xx} + P_1\delta\varepsilon^2_{xx} + R_3\delta\varepsilon^5_{zz} + N_5\delta\varepsilon^0_{xz}\\ &+ Q_5\delta\varepsilon^3_{xz} + K_5\delta\varepsilon^4_{xz} - q\delta\overline{w})dxdy, \end{aligned} \tag{11}$$

where σ or ε are the stresses and the strain vectors, q is the distributed transverse load; and N_i, M_i, P_i, Q_i, K_i and R_i are the resultants of the following integrations:

$$\begin{aligned} (N_i, M_i, P_i) &= \sum_{k=1}^{N}\int_{z^{(k-1)}}^{z^k} \sigma_i(1, z, f(z))dz, \quad (i=1),\\ N_i &= \sum_{k=1}^{N}\int_{z^{(k-1)}}^{z^k} \sigma_i dz, \quad (i=5),\\ (Q_i, K_i) &= \sum_{k=1}^{N}\int_{z^{(k-1)}}^{z^k} \sigma_i(g(z), f'(z))dz, \quad (i=5),\\ R_i &= \sum_{k=1}^{N}\int_{z^{(k-1)}}^{z^k} \sigma_i g'(z)dz, \quad (i=3), \end{aligned} \tag{12}$$