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Tolerance modelling of vibrations of periodic three-layered plates with inert core

Jakub Marczak *, Jarosław Jędrysiak

Department of Structural Mechanics, Łódź University of Technology, Al. Politechniki 6, 90-924 Łódź, Poland

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ABSTRACT

In this note a free vibration analysis of periodic three-layered sandwich structures is performed. Basing on the Kirchhoff's thin plate theory simplified equations of motion are derived, which are characterised by highly-oscillating, periodic and non-continuous coefficients. In order to obtain a system of equations with constant coefficients, the tolerance averaging technique is used. An application of the proposed tolerance model to analyse free vibration frequencies of a three-layered plate strip is shown – for both lower order frequencies related to its macrostructure and higher order frequencies related to its microstructure. Some comparisons of results of lower frequencies, obtained in the tolerance, the asymptotic and the known homogenised models are presented. Moreover, a certain verification of the proposed model is performed using the Ritz method. It can be observed that the tolerance model can be successfully applied to analyse vibration problems of vast variety of periodic three-layered plates and can significantly improve the optimisation process of such structures.

1. Introduction

There are many reasons why composite structures are becoming more and more vital for modern engineering. As state of the art technology allows us to combine several different materials into one heterogeneous structure, characterised by physical and mechanical properties, which are unreachable for classic materials, it becomes crucial to develop useful tool for optimising these properties for special engineering purposes.

In this article three-layered 'sandwich' composite structures are considered. Investigations on behaviour of such structures have their beginnings in the middle of 20th century and since then the design and optimisation process has been much improved. The analysis of dynamical behaviour of sandwich structures can be found in works of Chonan [1], Oniszczuk [2,3] and Szcześniak [4,5], among others. As it became clear that the shape of the core of sandwich structures is of great importance for mechanical properties of the whole structure, many researchers investigate this relation. Hence, many different concepts of periodic (e.g. honeycomb, rectangular, wavy-type, cf. Massimo and Panos [6]) or quasi-periodic (e.g. aluminium or metal foam, cf. Jin-Yih et al. [7], Magnucki and Jasion [8], Grygorowicz et al. [9], Jasion et al. [10]) cores have been presented and the need for a convenient mathematic model of periodic structures has emerged.

Some propositions to describe discrete periodic structures was performed by Brillouin [11], where the vibration analysis of atomic lattice was investigated. Basing on his work several discrete and continuous models of a wave propagation in periodic structures were proposed, e.g. by Mead [12]. A different approach was proposed by Kohn and Vogelius [13], who presented the homogenisation method for periodic plates, which special application was used recently e.g. by Wen-ming et al. [14]. However, governing equations of these methods usually neglect the influence of the microstructure on behaviour of these plates, which in certain engineering cases can prove to be insufficient.

With the development of computers, the finite element method (FEM) become one of the most popular approaches to analyze periodic structures. One should mention the work by Zhi-Jing et al. [15], which shows a vibration analysis of periodic plates using a spectral element method, being a special application of FEM, investigations of Massimo and Panos [6] of wave propagation in sandwich plates with periodic honeycomb core or numerical analysis of vibrations of periodic plates by Yuanwu et al. [16], using the asymptotic homogenisation method. Since the use of FEM for vibration analysis of various periodic structures is much time-consuming, different analytical solutions are proposed.

In this note the analytical solution to a vibration analysis of periodic three-layered structures with an inert core is presented

* Corresponding author.
E-mail addresses: jakub.marczak@p.lodz.pl (J. Marczak), jarek@p.lodz.pl (J. Jędrysiak).

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and discussed. Basing on the simplified model, shown by Szcześniak [4], governing equations of motion with coefficients being periodic, non-continuous and highly-oscillating functions are obtained. In order to derive a system of equations with constant coefficients, which take into consideration the effect of the microstructure on the behaviour of the whole structure, *the tolerance averaging technique*, proposed by Woźniak et al. [17,18], is applied. Eventually, the obtained solutions are compared to results by the asymptotic and the homogenised models. A certain physical correctness of the proposed model is also shown using the Ritz method.

2. Modelling foundations

Let $Ox_1x_2x_3$ be an orthogonal Cartesian coordinate system, t – a time coordinate and $\boldsymbol{x} \equiv (x_1, x_2)$. The considered structure is assumed to have spans L_1 and L_2 in x_1- and x_2-axis directions, respectively. Hence, its midplane is defined as $\Delta \equiv [0, L_1] \times [0, L_2]$. By setting $z \equiv x_3$ the undeformed plate occupies the region $\Lambda \equiv \{(\boldsymbol{x}, z) : -H(\boldsymbol{x})/2 \leqslant z \leqslant H(\boldsymbol{x})/2, \boldsymbol{x} \in \Delta\}$, where $H(\boldsymbol{x})$ is a total thickness of the plate.

The outer layers of the considered structure are Kirchhoff's type thin plates, which are assumed to be symmetric to the structure midplane Δ and are made of the same materials. Hence, all material and mechanical properties of these layers are identical. Let us introduce the following denotations, describing outer layer's bending stiffness B and mass density per unit area μ:

$$B_{\alpha\beta\gamma\delta} \equiv B^1_{\alpha\beta\gamma\delta}(\boldsymbol{x}) = B^2_{\alpha\beta\gamma\delta}(\boldsymbol{x}) = \int_{-h(\boldsymbol{x})/2}^{h(\boldsymbol{x})/2} C_{\alpha\beta\gamma\delta}(\boldsymbol{x}, z) z^2 \, dz,$$
$$\mu \equiv \mu^1(\boldsymbol{x}) = \mu^2(\boldsymbol{x}) = \int_{-h(\boldsymbol{x})/2}^{h(\boldsymbol{x})/2} \rho(\boldsymbol{x}, z) dz, \tag{1}$$

where $C_{\alpha\beta\gamma\delta}(\boldsymbol{x}, z)$ is elastic modulus tensor of the outer layers, ν is the Poisson's ratio, $h(\boldsymbol{x})$ is the thickness of the outer layer (cf. Fig. 1) and $\rho(\boldsymbol{x}, z)$ is mass density. An additional condition is that the outer plates are connected with each other by elastic core, characterised by certain elasticity modulus $k(\boldsymbol{x})$, thickness $h_c(\boldsymbol{x})$ and mass density $\rho_c(\boldsymbol{x})$ (cf. Fig. 1).

Considering the described structure of the sandwich plate, it is possible to distinguish a small, repeatable element, called *the periodicity cell*. Every cell has dimensions l_1 and l_2 in x_1 and x_2 direction, respectively. Hence it occupies the region $\Omega \equiv [-l_1/2, l_1/2] \times [-l_2/2, l_2/2]$. The diameter of the periodicity cell, given by: $l \equiv \left(l_1^2 + l_2^2\right)^{1/2}$, is called *the microstructure parameter* and must fulfil the following conditions: $h(\boldsymbol{x}) \ll l \ll \min(L_1, L_2)$.

Let us denote an overdot as a time derivative and ∂_α as a partial derivative with respect to a space coordinate. Taking into account above conditions, the equations of motion for the considered plate structure can be written as follows:

$$\partial_{\alpha\beta}(B_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}u_1) + \mu\ddot{u}_1 + k(u_1 - u_2) = f_1,$$
$$\partial_{\alpha\beta}(B_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}u_2) + \mu\ddot{u}_2 + k(u_2 - u_1) = f_2, \tag{2}$$

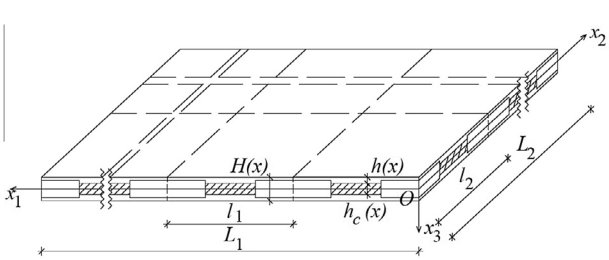

Fig. 1. The periodic three-layered plate.

where u_1, u_2 are deflections of upper and lower plate, respectively, and f_1, f_2 are loadings defined as:

$$f_1 \equiv p_1 - \frac{1}{2}\mu_c\ddot{u}_1, \quad f_2 \equiv p_2 - \frac{1}{2}\mu_c\ddot{u}_2, \quad \mu_c = \int_{-h_c(x)/2}^{h_c(x)/2} \rho_c(x, z) dz. \tag{3}$$

Eq. (2) are partial differential equations with periodic, highly-oscillating and non-continuous coefficients. Finding analytical solution to this system of equations is rather difficult and much time-consuming, hence it will be transformed, using *the tolerance averaging technique*, into equations with constant coefficients.

3. Basic concepts of the tolerance averaging technique

The tolerance averaging technique was described and developed by Woźniak et al. in a numerous books and publications [17,18]. Various applications of the method were presented in a series of papers, e.g. by Jędrysiak [19,20], Jędrysiak and Michalak [21,22] or Domagalski and Jędrysiak [23]. In the tolerance averaging technique, several introductory concepts are applied, defined below.

Denote a cell at $x \in \Lambda_\Omega$ by $\Omega(x) \equiv x + \Omega$, and $\Lambda_\Omega \equiv \Lambda \cap \cup_{z \in \Omega(x)} \Omega(z)$. The definition of *the averaging operation* for an integrable function f can be presented in the form:

$$\langle f \rangle(x) = \frac{1}{|\Omega|} \int_{\Omega(x)} f(y) dy, \quad x \in \Lambda_\Omega. \tag{4}$$

As a result of applying the averaging operation to a periodic function f in x, the constant averaged value of f is obtained.

Let us $\partial^k f$ be the kth gradient of function $f = f(x), x \in \Lambda$, $k = 0, 1, \ldots, \alpha$, $(\alpha \geqslant 0)$; $\partial^0 f \equiv f$. By $\widetilde{f}^{(k)}(\cdot, \cdot)$ denote a function defined in $\bar{\Lambda} \times R^m$, and by δ – a tolerance parameter. Let us also introduce $\Lambda_x \equiv \Lambda \cap \bigcup_{z \in \Omega(x)} \Omega(z)$, $x \in \bar{\Lambda}$.

Function $f \in H^\alpha(\Lambda)$ is called *the tolerance-periodic function*, $f \in TP^\alpha_\delta(\Lambda, \Omega)$, if for $k = 0, 1, \ldots, \alpha$, the following conditions are satisfied:

(1) $(\forall x \in \Lambda)\ (\exists \widetilde{f}^{(k)}(x, \cdot) \in H^0(\Omega))\ [||\partial^k f|\Lambda_x(\cdot) - \widetilde{f}^{(k)}(x, \cdot)||_{H^0(\Lambda_x)} \leqslant \delta]$,
(2) $\int_{\Omega(\cdot)} \widetilde{f}^{(k)}(\cdot, z) dz \in C^0(\bar{\Lambda})$.

Function $\widetilde{f}^{(k)}(x, \cdot)$ is called *the periodic approximation of $\partial^k f$ in* $\Omega(x), x \in \Lambda$, $k = 0, 1, \ldots, \alpha$,.

Function $F \in H^\alpha(\Lambda)$ is called *the slowly-varying function*, $F \in SV^\alpha_\delta(\Lambda, \Omega)$, if:

(1) $F \in TP^\alpha_\delta(\Lambda, \Omega)$,
(2) $(\forall x \in \Lambda)\ [\widetilde{F}^{(k)}(x, \cdot)|_{\Omega(x)} = \partial^k F(x), \quad k = 0, \ldots, \alpha]$.

Function $\phi \in H^\alpha(\Lambda)$ is called *the highly oscillating function*, $\phi \in HO^\alpha_\delta(\Lambda, \Omega)$, if:

(1) $\phi \in TP^\alpha_\delta(\Lambda, \Omega)$,
(2) $(\forall x \in \Lambda)\ [\widetilde{\phi}^{(k)}(x, \cdot)|_{\Omega(x)} = \partial^k \widetilde{\phi}(x), \quad k = 0, 1, \ldots, \alpha]$,
(3) $\forall\, F \in SV^\alpha_\delta(\Lambda, \Omega) \quad \exists f \equiv \phi F \in TP^\alpha_\delta(\Lambda, \Omega)$
$\widetilde{f}^{(k)}(x, \cdot)|_{\Omega(x)} = F(x) \partial^k \widetilde{\phi}(x)|_{\Omega(x)}, \quad k = 1, \ldots, \alpha.$

For $\alpha = 0$ let us denote $\widetilde{f} \equiv \widetilde{f}^{(0)}$.

Let $g(\cdot)$ be defined on $\bar{\Lambda}$ as a highly-oscillating function, $g \in HO^2_\delta(\Lambda, \Omega)$, continuous together with gradient $\partial^1 g$. However, gradient $\partial^2 g$ is a piecewise continuous and bounded. Function g