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Singular integrals in boundary elements for coupled stretching–bending analysis of unsymmetric laminates

Chyanbin Hwu *, H.W. Chang [1]

Institute of Aeronautics and Astronautics, National Cheng Kung University, Tainan, Taiwan, ROC

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ABSTRACT

When an unsymmetric laminated plate is considered, the coupling of in-plane and plate bending problems is unavoidable. The boundary element for the coupled stretching–bending analysis was developed previously with suitable complex form fundamental solution. Usually, the singular integrals involved in the boundary element are treated by the conventional methods such as Gaussian quadrature rule for regular functions, logarithmic Gaussian quadrature formulas for the function with logarithmic terms, and the use of finite part integrals for the evaluation in sense of Cauchy principal value, or calculated indirectly through the employment of rigid body movement. To avoid the complexity of the numerical integration with complex form fundamental solution, in this paper we provide the explicit closed-form solutions for the singular integrals, which simplify the computer programming and expedite the numerical computation. And hence, the accuracy and efficiency of the associated boundary elements are improved through the newly derived analytical solutions.

1. Introduction

In practical applications, to take advantage of the designable characteristics of composite laminates, it always has the possibility to design a plate with unsymmetric laminated composites. In that case, the stretching–bending coupling may occur no matter what kind of loading is applied on the laminated plates. Due to the coupling of stretching and bending deformations, the stress analysis of the unsymmetric laminated plates become much more complicate than that of the metallic plates or the symmetric laminated plates since the latter can be treated by considering only in-plane or plate bending analysis. To effectively treat the coupled stretching–bending deformation, a boundary integral equation (BIE) was derived using the reciprocal theorem of Betti and Rayleigh, and its associated fundamental solution was derived using the Stroh-like complex variable formalism [1]. In numerical programming it was found that some integrals of BIE derived in [1] will become singular when the field point approaches to the source point, and hence a modified BIE and boundary element was proposed later [2] by separating the integrals into two parts: singular part and regular part.

Generally, the integrals involved in the boundary elements may be categorized into regular integrals, weakly singular integrals, strongly singular integrals, and hypersingular integrals [3]. The numerical integration of the regular integral is straightforward and can be carried out using standard Gaussian quadrature rule. The singular integrals with the forms of $\ln r$ and $1/r$ where r is the distance from a source point to a field point can be evaluated via logarithmic Gaussian quadrature formulas [4], direct computation of the integrals [5,6] or the use of finite part integrals [3,4,7–11]. The discussion about the concepts of Cauchy principal value and finite part integrals can be found in [12–18]. Suitable employment of rigid body movements may also provide an indirect result related to the singular integrals [5,19,20].

Although the singular integrals are common problems for boundary element method and their solution techniques are well documented, most of them are restricted to two-dimensional or three-dimensional analysis with the fundamental solutions expressed in real form. Very few of them discussed the coupling between in-plane and plate bending problems with complex form fundamental solutions. Since the fundamental solutions embedded in the boundary element analysis for unsymmetric laminated composites are written in complex variable matrix functions, at the first glance it is not that direct to use the classical methods to deal with the singular integrals. For example, the logarithmic function with real variable is only valid for the positive argument, whereas the same function with complex variable is valid for any complex

* Corresponding author. Tel.: +886 6 2757575; fax: +886 6 2389940.
E-mail address: CHwu@mail.ncku.edu.tw (C. Hwu).
[1] Tel.: +886 6 2757575; fax: +886 6 2389940.

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number (including negative real number) and to have a unique value a suitable branch cut should be selected for the principal value of the logarithmic function. Due to the selection of the branch cut for a unique function value, discontinuity may occur at the cut region. And hence, to satisfy the continuity requirement for a continuum a special attention should be made on the numerical programming with complex function. This problem will not occur in the real form fundamental solution, but now for the coupled stretching–bending analysis it happens if the available fundamental solutions are expressed in complex form.

To deal with the complex function singular integrals, the fundamental solutions obtained for the coupled stretching–bending analysis were re-derived in this paper to make the singular integrals expressed in real function. Within each element the deflection is interpolated by a third order polynomial, and all the other quantities such as boundary geometry and in-plane displacements, normal slopes, and the tractions, are interpolated by a linear function. With the newly derived fundamental solutions and the selected interpolation functions, the analytical solutions of the singular integrals involved in the influence matrices of the boundary elements for the coupled stretching–bending analysis are obtained in this paper.

To verify the correctness of our derived analytical solutions, comparison was made with the Cauchy principal value [12] calculated by using the technique of finite part integrals. An alternative verification was also made through the free term coefficients calculated via the use of rigid body movements. Furthermore, to show its applicability to the general problems of laminated plates, a simply supported unsymmetric laminate [0/45/90/30/−45/90/45/−60] with a rectangular cut-out subjected to a uniform lateral load is illustrated. The results of deformation and stresses calculated by the present boundary element are shown to be well agreed with those obtained by the commercial finite element software ANSYS.

2. Stroh-like formalism for coupled stretching–bending theory

In classical lamination theory, the equations of displacement fields, strain–displacement relations, constitutive laws and equilibrium can be written as follows:

$$u = u_0 - z\frac{\partial w}{\partial x}, \quad v = v_0 - z\frac{\partial w}{\partial y}, \quad w = w_0, \tag{2.1a}$$

$$\begin{aligned}&\varepsilon_x = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_y = \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w}{\partial y^2},\\ &\gamma_{xy} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y},\end{aligned} \tag{2.1b}$$

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}, \tag{2.1c}$$

$$\begin{aligned}&\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} = 0, \quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0,\\ &\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0\end{aligned} \tag{2.1d}$$

in which $u,\ v$ and w are the displacements in $x,\ y$ and z directions; $u_0,\ v_0$ and w_0 are the middle surface displacements; $(\varepsilon_x, \varepsilon_y, \gamma_{xy})$ are the strains, $(\varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0) = (\partial u_0/\partial x, \partial v_0/\partial y, \partial u_0/\partial y + \partial v_0/\partial x)$ are the mid-plane strains and $(\kappa_x, \kappa_y, \kappa_{xy}) = (-\partial^2 w/\partial x^2, -\partial^2 w/\partial y^2, -2\partial^2 w/\partial x \partial y)$ are the plate curvatures; $N_x,\ N_y,\ N_{xy}$ and $M_x,\ M_y,\ M_{xy}$ are the stress resultants and bending moments; q is the lateral distributed load applied on the laminates; $A_{ij},\ B_{ij}$ and D_{ij} are, respectively, the extensional, coupling and bending stiffnesses [21].

If the coupling stiffness B_{ij} is non-zero, which usually occurred in unsymmetric laminates, the plates will be stretched as well as bent even under pure in-plane forces or pure bending moments. Due to the coupling effects, the coupled stretching–bending analysis becomes much more complicated than the two-dimensional analysis and the plate bending analysis. An elegant and powerful complex variable method called Stroh-like formalism was developed to deal with this kind of problems. With this formalism, a general solution satisfying all the basic Eqs. (2.1a–d) with $q = 0$ can be expressed as [1,22]

$$\mathbf{u}_d = 2Re\{\mathbf{Af}(z)\}, \quad \boldsymbol{\phi}_d = 2Re\{\mathbf{Bf}(z)\}, \tag{2.2a}$$

where

$$\begin{aligned}&\mathbf{u}_d = \begin{Bmatrix} \mathbf{u} \\ \boldsymbol{\beta} \end{Bmatrix}, \quad \boldsymbol{\phi}_d = \begin{Bmatrix} \boldsymbol{\phi} \\ \boldsymbol{\psi} \end{Bmatrix},\\ &\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4], \quad \mathbf{B} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4],\\ &\mathbf{f}(z) = \begin{Bmatrix} f_1(z_1) \\ f_2(z_2) \\ f_3(z_3) \\ f_4(z_4) \end{Bmatrix}, \quad z_k = x_1 + \mu_k x_2, \quad k = 1,2,3,4\end{aligned} \tag{2.2b}$$

and

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}, \quad \boldsymbol{\beta} = \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix}, \quad \boldsymbol{\phi} = \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix}, \quad \boldsymbol{\psi} = \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix} \tag{2.2c}$$

in which Re denotes the real part of a complex number; μ_k and $(\mathbf{a}_k, \mathbf{b}_k),\ k = 1,2,3,4$ are, respectively, the *material eigenvalues and material eigenvectors* related to extensional, coupling and bending stiffnesses; $f_k(z_k),\ k = 1,2,3,4$, are the analytical complex function determined by the boundary conditions set for the problem. In Eq. (2.2c), u_1 and u_2 are the mid-plane displacements u_0 and v_0; $\beta_1 = -\partial w/\partial x$, $\beta_2 = -\partial w/\partial y$, are the negative slopes in x and y directions; ϕ_1 and ϕ_2 are the stress functions related to the in-plane forces N_{ij}, and ψ_1 and ψ_2 are the stress functions related to the bending moments M_{ij}, transverse shear forces Q_i and effective transverse shear forces V_i. Their relations are

$$\begin{aligned}&N_{11} = -\phi_{1,2}, \quad N_{22} = \phi_{2,1}, \quad N_{12} = \phi_{1,1} = -\phi_{2,2} = N_{21},\\ &M_{11} = -\psi_{1,2}, \quad M_{22} = \psi_{2,1}, \quad M_{12} = \psi_{1,1} - \eta = -\psi_{2,2} + \eta = M_{21},\\ &Q_1 = -\eta_{,2}, \quad Q_2 = \eta_{,1}, \quad \eta = (\psi_{1,1} + \psi_{2,2})/2,\\ &V_1 = -\psi_{2,22}, \quad V_2 = \psi_{1,11}\end{aligned} \tag{2.3}$$

in which the subscript $\bullet_{,i},\ i = 1,2$ denotes differentiation with respect to x_i.

2.1. Green's function for laminates

Consider an infinite laminate subjected to a concentrated force $\hat{\mathbf{f}} = (\hat{f}_1, \hat{f}_2, \hat{f}_3)$ and moment $\hat{\mathbf{m}} = (\hat{m}_1, \hat{m}_2, \hat{m}_3)$ at point $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2)$ (see Fig. 1). The solution to this problem, which is usually called *Green's function*, has been obtained by using the Stroh-like formalism and can be written in the form of (2.2a) in which the complex function vector $\mathbf{f}(z)$ is [23]