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Coupled higher order and mixed layerwise finite element based static and free vibration analyses of laminated plates

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ABSTRACT

A unique finite element model for static and free vibration analyses of thick and thin composite laminates is presented. The model is a combination of 3D mixed layerwise and equivalent single layer (ESL) theories. The ESL is employed in the global part of domain. On the other hand, a stack of 3D elements are used for estimation of the local parameters. The transverse inter laminar stresses are determined by using 18 noded 3D mixed layerwise element with 6 DOF per node in the local region. This mixed element incorporates the interlaminar stresses as the nodal DOF in addition to displacements for ensuring continuity of the transverse stresses in the thickness direction. Nine noded 2D elements with 12 DOF per node are used in the global domain. A transition has been developed for connection and compatibility of differently modelled sub-domains. Hamilton's variational principle has been used for the free vibration analysis. Present static and vibration analyses of laminates are in good agreement with the available elasticity and closed form solutions. The presented combined mesh modelling reduces number of elements to map the entire domain as compared to full 3D model. This results in substantial reduction of DOF and improves the computational efficiency.

1. Introduction

Laminated composites with continuous fibres are widely being used in various structural applications. Composites are preferred due to their characteristics like high stiffness to weight ratio, good resistance to corrosive agents and capability of being engineered according to requirements.

Analysis of laminated composites is a complex phenomenon due to heterogeneous material properties of different layers. ESL and Layerwise theories (LWT) are often employed for the analysis of composite laminates. These theories have their own merits and shortcomings. For example, ESL formulations are computationally less expensive and predict the global parameters with reasonable accuracy but fail to ensure continuity of the transverse stresses. On the other hand, LWT are more accurate but are computationally expensive due to presence of a relatively higher DOF.

Various modifications have been proposed to overcome drawbacks of ESL. For example, Mau et al. [1] developed multilayer stress-based FEs based on Pian's assumed stress hybrid method [2]. On the other hand, Kant and Swaminathan [3,4] proposed a higher order theory based on the 3D elasticity concepts with a complete cubic displacement field with 12 DOF(HOST12) derived from the Taylor series which encompasses all possible inplane and transverse deformations. Evaluation of the transverse stresses, however, requires integration of stress equilibrium equations.

Elasticity solutions for layered plates [5–7] indicate that interlaminar continuity of the transverse normal and shear stresses as well as the layer-wise continuous displacement field through the thickness of laminated plates are essential requirements. Thus, a layer-wise analysis is often required for laminated composite structures. Various displacement based LWT and FE models have been proposed by Soldatos [8], Wu and Kuo [9], and others. These displacement based LW formulations have been reported to provide satisfactory results for the global values (e.g. deflections and flexural stresses) as well as for the local values (e.g. the transverse stresses) for thin as well as thick laminates. However, continuity of the transverse stress components at the interface cannot be enforced, although continuity of displacement field through the thickness can be satisfied.

A layer-wise mixed finite element model with displacement and the transverse stress components as primary variables can very well satisfy requirements of the transverse stress continuity in addition to continuity of displacement fields through the thickness of laminated composites. Transverse stress components are evaluated directly through such mixed FE model. Thus, integration of equilibrium equations, which involve differentiation of in-plane stresses and displacement fields, can be advantageously avoided.

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Exhaustive literature on mixed finite element models have been compiled by Noor [10]. Wu and Lin [11], for example, presented a two-dimensional mixed finite element scheme based on a local high-order displacement model for analysis of sandwich structure, where displacement continuity conditions at the interface between layers were regarded as constraints and the interlaminar stresses were introduced as Lagrange multiplier. Shi and Chen [12], on the other hand, developed a three-dimensional mixed FE model based on global–local laminate variational model. Model proposed a mixed use of a hybrid stress element within a high precision stress solution region in the thickness direction of laminate and a conventional displacement finite element in remaining region.

Carrera [13] has done work on development of mixed finite element model. Carrera et al. [14,15], Carrera and Petrolo [16], Carrera and Miglioretti [17], Mashat et al. [18] and Cinfera et al. [19] proposed a unified formulation in which displacement field in a laminate is defined in concise form and can be expanded using different series solutions. The number of terms or presence of any term in the displacement field can be controlled to evolve a different theory for the analysis of laminates. A best plate theory based on error analysis on various combinations of the terms considered in displacement field is suggested. The unified formulations have been proposed for ESL and also for LW analyses. Murakami's zigzag function has also been incorporated in the unified formulations.

Ramtekkar et al. [20] developed a mixed finite element model with the transverse stresses included in the set of nodal DOF along with displacements. Formulation was extended by Desai et al. [21] for dynamic analysis of composite plates. Recently, Sahoo and Singh [22] presented an inverse hyperbolic zigzag theory which satisfies traction free conditions and transverse stress continuity. Many researchers [23–28] have presented formulations for free and forced vibrations of composite laminates.

Literature indicates that layer wise approach is desirable for accurate estimation of the interlaminar stresses. Amongst the layer wise models reported in the literature, it is observed that the mixed formulation [20] having the transverse stresses and the nodal deformations as the degrees of freedom yields very accurate solution and also satisfies continuity requirements coherently. However, such formulation is computationally expensive. Thus, layerwise formulation may not be suitable at all locations in the entire domain of a laminate. Amongst ESL formulations, the higher order theory [3] with 12 DOF(HOST12) has been reported to be a comprehensive and computationally economical theory for estimation of global parameters.

In this work, the positive traits of higher order ESL theory and mixed LWT are employed in tandem to obtain global parameters as well as to capture the local transverse stress parameters with a reasonable accuracy. Laminate is modelled using a stack of 3D elements having mixed DOF (displacements and transverse stresses) in zones where the transverse stresses are critical. Remaining zone is modelled by using the ESL elements based on a displacement field fully cubic in the thickness direction(HOST12). Transition elements are used at the interface of ESL and mixed LWT based elements.

2. Theoretical formulation

Three models have been formulated for analysis of transversely loaded laminated composite plates consisting of several orthotropic laminae.

(a) **Model 1:** This model adopts a cubic displacement field in the thickness direction for displacements (U,V,W) and has 12 DOF. The theory has been identified as HOST12. The model is based on the three dimensional state of stresses and strains.

(b) **Model 2:** In this model, mixed finite element LWT, which has three displacements (U,V,W) and the transverse stresses $(\tau_{xz}, \tau_{yz}, \sigma_z)$ as the nodal DOF, is used. The theory is based on elasticity relationships. Therefore, introduction of any additional parameters/stress variation functions are advantageously avoided.

(c) **Model 3:** This model is based on a local global finite element procedure to take advantage of computational efficiency of the higher order ESL theory and accuracy of the 3D mixed model.

2.1. MODEL 1: Development of ESL theory based model (HOST 12)

Displacements in three principal directions of the laminate as a fully cubic function of the thickness co-ordinate are

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + z\theta_x(x,y) + z^2 u_0^*(x,y) + z^3\theta_x^*(x,y) \\ v(x,y,z) &= v_0(x,y) + z\theta_y(x,y) + z^2 v_0^*(x,y) + z^3\theta_y^*(x,y) \\ w(x,y,z) &= w_0(x,y) + z\theta_z(x,y) + z^2 w_0^*(x,y) + z^3\theta_z^*(x,y) \end{aligned} \tag{1}$$

The above displacement field eliminates any requirement of shear correction factor and chances of shear locking. Here, u_0, v_0 and w_0 are the deformations in the x, y, z (laminate co-ordinate) directions, respectively, at the midplane. θ_x, θ_y and θ_z, on the other hand, are the rotations at midplane about the principal directions of laminate. $u_0^*, \theta_x^*, v_0^*, \theta_y^*, w_0^*$ and θ_z^* are higher order terms stemming from the Taylor's series. By using material property, the strain displacement relationship and the principle of minimum potential energy, the stiffness matrix for laminate is developed. By using shape functions similar to the stiffness evaluation, the mass matrix is also developed. Detailed formulation can be seen in [3]. A nine noded Lagrangian isoparametric element has been used to discretize a laminate.

Numerical integration is performed by employing 3×3 Gauss quadrature rule for the extension, bending, mass component whereas 2×2 Gauss rule for the shear part.

2.2. MODEL 2: Development of mixed model

An 18-node three-dimensional element based on mixed formulation is used by considering displacement fields $u(x,y,z)$, $v(x,y,z)$ and $w(x,y,z)$ having quadratic variation along the plane of plate and cubic variation in the transverse direction. The cubic variation of field has been adopted to invoke the transverse stresses as the nodal parameters in addition to the nodal deformations. The displacement field is expressed as

$$u_k(x,y,z) = \sum_{i=1}^{3}\sum_{j=1}^{3} g_i h_j a_{0ijk} + z\sum_{i=1}^{3}\sum_{j=1}^{3} g_i h_j a_{1ijk} + z^2\sum_{i=1}^{3}\sum_{J=1}^{3} g_i h_j a_{2ijk} + z^3\sum_{i=1}^{3}\sum_{j=1}^{3} g_i h_j a_{3ijk} \tag{2}$$

where

$$g_1 = \frac{\xi}{2}(\xi - 1), g_2 = 1 - \xi^2, \quad g_3 = \frac{\xi}{2}(1+\xi), \xi = x/L_x$$

$$h_1 = \frac{\delta}{2}(\delta - 1), \quad h_2 = 1 - \delta^2, \quad h_3 = \frac{\delta}{2}(1+\delta), \quad \delta = y/L_y$$

$k = 1, 2, 3$ and $\quad u_1 = u;\ u_2 = v;\ u_3 = w;$

Further, a_{mijk} (m = 0, 1, 2, 3; i, j, k = 1, 2, 3) are the generalised coordinates.

Variation of displacement fields has been assumed to be cubic through the thickness of element, although there are only two nodes along 'z' axis of an element. Derivative of displacement with