Composite Structures 126 (2015) 227–232

Contents lists available at ScienceDirect

Composite Structures

journal homepage: www.elsevier.com/locate/compstruct

Convergence theorem for the Haar wavelet based discretization method

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ARTICLE INFO

Article history:
Available online 25 February 2015

Keywords:
Haar wavelet method
Accuracy issues
Convergence theorem
Numerical evaluation of the order of convergence
Extrapolation

ABSTRACT

The accuracy issues of Haar wavelet method are studied. The order of convergence as well as error bound of the Haar wavelet method is derived for general nth order ODE. The Richardson extrapolation method is utilized for improving the accuracy of the solution. A number of model problems are examined. The numerically estimated order of convergence has been found in agreement with convergence theorem results in the case of all model problems considered.

1. Introduction

Nowadays, the Haar wavelets are most widely used wavelets for solving differential and integro-differential equations, outperforming Legendre, Daubechie, etc. wavelets (Elsevier scientific publication statistics). Prevalent attention on Haar wavelet discretization methods (HWDM) can be explained by their simplicity. The Haar wavelets are generated from pairs of piecewise constant functions and can be simply integrated. Furthermore, the Haar functions are orthogonal and form a good transform basis.

Obviously, the Haar functions are not differentiable due to discontinuities in breaking points. As pointed out in [1] there are two main possibilities to overcome latter shortcomings. First, the quadratic waves can be regularized ("smoofed") with interpolating splines, etc. [2,3]. Secondly, an approach proposed by Chen and Hsiao in [4,5], according to which the highest order derivative included in the differential equation is expanded into the series of Haar functions, can be applied. Latter approach is applied successfully for solving differential and integro-differential equations in most research papers covering HWDM [1,4–28]. Following the pioneering works Chen and Hsiao in [4,5] Lepik developed the HWDM for solving wide class of differential, fractional differential and integro-differential equations covering problems from elastostatics, mathematical physics, nonlinear oscillations, evolution equations [1,6–10]. The results are summarized in monograph [11]. It is pointed out by Lepik in [1,11] that the HWDM is convenient for solving boundary value problems, since the boundary conditions can be satisfied automatically (simple analytical approach).

Composite structures are examined by use of wavelets first in [12,2]. In [12] the free vibration analysis of the multilayer composite plate is performed by adapting HWDM. The static analysis of sandwich plates using a layerwise theory and Daubechies wavelets is presented in [2]. The delamination of the composite beam is studied in [13]. During last year Xiang et al. adapted HWDM for free vibration analysis of functionally graded composite structures [14–18]. In [14–18] a general approach for handling boundary conditions has been proposed. In all above listed studies the Haar wavelet direct method is applied. The weak form based HWDM has been developed in [19], where the complexity analysis of the HWDM has been performed. Recent studies in area of wavelet based discretization methods cover solving fractional partial differential equations by use of Haar, Legendre and Chebyshev wavelets [20–25]. In [26–28] the Haar wavelets are utilized for solving nuclear reactor dynamics equations. The neutron point kinetics equation with sinusoidal and pulse reactivity is studied in [26]. In [27,28] are solved neutron particle transport equations. In [29–31] the HWDM is employed with success for solving nonlinear integral and integro-differential equations.

Most of papers overviewed above found that the implementation of the HWDM is simple. Also, the HWDM is characterized most commonly with terms „simple", "easy" and effective" (see [1,14–18,25–28] and others). The review paper [32] concludes that the HWDM is efficient and powerful in solving wide class of linear and nonlinear reaction–diffusion equations.

However, no convergence rate proof found in literature for this method. It is shown in several papers [33–35] that in the case of

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http://dx.doi.org/10.1016/j.compstruct.2015.02.050
0263-8223/

function approximation with direct expansion into Haar wavelets, the convergence rate is one. This result hold good for function approximation in integral equations, but does not hold good for HWDM developed for differential equations above, since in these methods instead of the solution its higher order derivative is expanded into wavelets.

The aim of the current study is to clarify the accuracy issues of the HWDM, based on approach introduced by Chen and Hsiao in [4,5], and featured for solving general nth order ordinal differential equations (ODE). This question is open from 1997 up to now. Answer to this question allows to give scientifically founded estimate to HWDM, also to make comparisons with other methods.

2. Haar wavelet family

In the following the Haar wavelet family is defined by using notation introduced by Lepik [1]. Let us assume that the integration domain $[A,B]$ is divided into $2M$ equal subintervals each of length $\Delta x = (B-A)/(2M)$. The maximal level of resolution J is defined as $M = 2^J$. The Haar wavelet family $h_i(x)$ is defined as a group of square waves with magnitude ± 1 in some intervals and zero elsewhere

$$h_i(x) = \begin{cases} 1 & \text{for} \quad x \in [\xi_1(i), \xi_2(i)), \\ -1 & \text{for} \quad x \in [\xi_2(i), \xi_3(i)), \\ 0 & \text{elsewhere}, \end{cases} \tag{1}$$

where

$$\xi_1(i) = A + 2k\mu\Delta x, \quad \xi_2(i) = A + (2k+1)\mu\Delta x,$$
$$\xi_3(i) = A + 2(k+1)\mu\Delta x, \quad \mu = M/m, \quad \Delta x = (B-A)/(2M). \tag{2}$$

In Eqs. (1) and (2) $j = 0,1,\ldots,J$ and $k = 0,1,\ldots,m-1$ stand for dilatation and translations parameters, respectively. The index i is calculated as $i = m+k+1$. Each Haar function contains one square wave, except scaling function $h_1(x) \equiv 1$. The parameter $m = 2^j$ $(M = 2^J)$ corresponds to a maximum number of square waves can be sequentially deployed in interval $[A,B]$ and the parameter k indicates the location of the particular square wave. Since the scaling function $h_1(x) \equiv 1$ does not include any waves here $m = 0,\ \xi_1 = A,\ \xi_2 = \xi_3 = B$. The Haar functions are orthogonal to each other and form a good transform basis

$$\int_0^1 h_i(x) h_l(x) dt = \begin{cases} 2^{-j} & i = l = 2^j + k, \\ 0 & i \neq l. \end{cases} \tag{3}$$

Any function $f(x)$ that is square integrable and finite in the interval $[A,B]$ can be expanded into a Haar wavelets as

$$f(x) = \sum_{i=1}^{\infty} a_i h_i(x). \tag{4}$$

The Haar coefficients

$$a_i = 2^j \int_A^B f(x) h_i(x) dx, \quad i = 1,\ldots,2^j + k + 1 \tag{5}$$

can be determined from minimum condition of integral square error as

$$\int_A^B E_M^2 dx \to \min, \quad |E_M| = |f(x) - f_M(x)|, \quad f_M(x) = \sum_{i=0}^{2M} a_i h_i(x). \tag{6}$$

In Eq. (5) $f(x)$ and $f_M(x)$ stand for the exact and approximate solutions, respectively. The integrals of the Haar functions (1) of order n can be calculated analytically as [1]

$$p_{n,i}(x) = \begin{cases} 0 & \textit{for} \quad x \in [A, \xi_1(i)), \\ \frac{(x-\xi_1(i))^n}{n!} & \text{for} \quad x \in [\xi_1(i), \xi_2(i)), \\ \frac{(x-\xi_1(i))^n - 2(x-\xi_2(i))^n}{n!} & \text{for} \quad x \in [\xi_2(i), \xi_3(i)), \\ \frac{(x-\xi_1(i))^n - 2(x-\xi_2(i))^n + (x-\xi_3(i))^n}{n!} & \text{for} \quad x \in [\xi_3(i), B). \end{cases} \tag{7}$$

Note that the integrals of the Haar functions are continuous functions in interval $[A,B]$. Also, the first integrals of the Haar functions are triangular functions $(\alpha = 1)$.

3. Convergence analysis of Haar wavelet discretization method

Let us consider nth order ordinal differential equation (ODE) in general form

$$G(x, u, u', u'', \ldots u^{(n-1)}, u^{(n)}) = 0, \tag{8}$$

where prime stand for derivative with respect to x. According to most commonly used approach introduced in [4,5] instead of solution of the differential equation its higher order derivative is expanded into Haar wavelets

$$f(x) = \frac{d^n u(x)}{dx^n} = \sum_{i=1}^{\infty} a_i h_i(x). \tag{9}$$

Using notation introduced in previous section, the sum in Eq. (9) can be rewritten as

$$f(x) = a_1 h_1 + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} a_{2^j+k+1} h_{2^j+k+1}(x). \tag{10}$$

In Eqs. (9) and (10) $i = m+k+1,\ j = 0,1,\ldots,J,\ k = 0,1,\ldots,m-1, m = 2^j$ $(M = 2^J)$. By integrating relation (9) n times one obtains the solution of the differential equation (8) as

$$u(x) = \frac{a_1 (B-A)^n}{n!} + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} a_{2^j+k+1} p_{n,2^j+k+1}(x) + B_T(x). \tag{11}$$

In Eq. (11) $B_T(x)$ and $p_{n,2^j+k+1}(x)$ stand for boundary term and nth order integrals of the Haar functions are determined by formula (7), respectively.

Without loss of generality it can be assumed in the following that $A = 0,\ B = 1$ since the differential equations can be converted into non-dimensional form by use of transform $\tau = (x-A)/(B-A)$ i.e. $x = A + (B-A)\tau$ (see [19]).

Theorem 1. *Let us assume that* $f(x) = \frac{d^n u(x)}{dx^n} \in L^2(R)$ *is a continuous function on* $[0,1]$ *and its first derivative is bounded*

$$\forall x \in [0,1] \quad \exists \eta : \left| \frac{df(x)}{dx} \right| \leqslant \eta, \quad n \geqslant 2 \ (\textit{boundary value problems}). \tag{12}$$

Then the Haar wavelet method, based on approach proposed in [4,5], will be convergent i.e. $|E_M|$ *vanishes as* J *goes to infinity. the convergence is of order two*

$$\|E_M\|_2 = O\left[\left(\frac{1}{2^{J+1}}\right)^2\right]. \tag{13}$$

Proof. It implies from Eqs. (5), (6) and (10) The error at the Jth level resolution can be written as

$$|E_M| = |u(x) - u_M(x)| = \left| \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} a_{2^j+k+1} p_{n,2^j+k+1}(x) \right|. \tag{14}$$