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Analytical solution for the three-dimensional stress fields in anisotropic composite bimaterial corners

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ABSTRACT

The paper presents a theoretical treatise for the three-dimensional elastic stress distributions close to anisotropic bimaterial corners according to which the 3D governing equations are successfully reduced to two uncoupled equations in the two-dimensional space. The former provides the solution of the corresponding plane notch problem, the latter provides the solution of the corresponding out-of-plane shear notch problem. With the new theory an analytical solution for the stress fields for bimaterial corners in thick anisotropic composite plates is presented and its degree of accuracy is discussed comparing theoretical results and numerical data from 3D FE analyses carried out on bimaterial composite plates and joints.

1. Introduction

For several decades scientists have put effort into the problem of finding stress distributions near points of singularity, basically with the aim to develop engineering strength criteria implicitly, or explicitly, based on local stress fields (see [1–6] and references reported therein).

Isotropic plates with angular corners were analysed by Williams [7] who showed that the near tip stress fields are governed by the term $r^{1-\lambda}$, where λ is a real number depending on the V-notch opening angle and r the distance from the V-notch tip. Williams analysis for isotropic materials was later extended to cracked anisotropic materials by Sih et al. [8], where it was proved that the elastic stress and strain singularity remains 0.5, and to wedge problems in anisotropic plates by Kuo and Bogy [9–11]. Following the same tracks many other authors dealt with this topic [12–18].

The eigen-function expansion method used in Ref. [7], was also adopted by Williams [19] to obtain the solution for dissimilar materials with a semi-infinite crack, where it was discovered for the first time that stresses can possess an oscillatory nature. Williams analysis was later extended to bending loads by Sih and Rice [20], and to any opening angle in the presence of in-plane loadings by Bogy [21,22], Demsey and Sinclair [23,24] and many other authors [25–28].

Due to the increasing use of advanced fibre reinforced composite materials, a large bulk of work was carried out also considering anisotropic multimaterial wedges subjected to antiplane deformation [29] or in-plane loadings (see, for example, [30–36] and references reported therein).

All the above mentioned papers are based on analyses carried out in the two-dimensional space. Instead, Pageau and Biggers [37] developed a three-dimensional finite element formulation to determine the order and angular variation of singular stress states due to material and geometric discontinuities in anisotropic materials. An eigenfunction expansion technique was used by Chaudhuri and Yoon [38,39] to study three-dimensional asymptotic displacement and stress fields in three-material plates. Yosibash [40,41], considering three-dimensional anisotropic domains in the vicinity of edge singularities, developed an innovative method for determining the edge eigen-pairs numerically with the aid of the finite element method.

Thanks to a number of 3D numerical analyses, Nakamura [42] investigated the stress fields near an interface crack in three-dimensional bimaterial plates and, for the first time, found out that, unlike the homogeneous case, the asymptotic field always consists of all three modes of fracture and that a significant antiplane (Mode III) deformation exists along the crack front, especially near the free surface. The phenomenon of coupled modes of fracture in three-dimensional homogeneous plates has been object of an extensive investigation in the recent years. Considering thick homogeneous and isotropic plates, Kotousov [43–46] developed an analytical treatise which made it evident how an out-of-plane shear stress singularity always exists, in addition to Williams' in-plane singularities. The intensity of local antiplane stress fields at the tip of pointed V-notches in 3D plates under remote Mode II loading was later discussed on the basis of detailed three-dimensional FE analyses [47–49].

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A new three-dimensional theory to be applied to thick anisotropic plates was developed by the present authors [50], where the 3D governing equations were reduced to two uncoupled equations in the two-dimensional space. Based on the new theory an analytical solution for the three-dimensional stress fields in homogeneous anisotropic composite plates with V-notches was presented and its degree of accuracy discussed versus 3D FE analyses.

In the present paper a theoretical treatise for the three-dimensional elastic stress distributions close to anisotropic composite bimaterial corners is presented. Based on the previous analytical treatment for homogeneous materials [50], the 3D governing equations are reduced to two uncoupled equations in the two-dimensional space, the former providing the solution of the corresponding plane notch problem, the latter the solution of the corresponding out-of-plane shear notch problem.

With the new theory the explicit analytical expressions of the 3D stress fields for bimaterial corners in thick anisotropic composite plates are derived. Eventually, considering relevant geometries, the degree of accuracy of the analytical solution is discussed comparing theoretical results and numerical data from 3D FE analyses.

2. The three-dimensional anisotropic elasticity problem

In this work materials are supposed to obey to a rectilinearly anisotropic elastic behaviour, according to which the in-plane and the antiplane problems are, by the material point of view, uncoupled [8,51]. Conventional fibre reinforced composite materials respect this hypothesis. Under this circumstance, stress–strain relationships can be written as:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ . & S_{22} & S_{23} & 0 & 0 & S_{26} \\ . & . & S_{33} & 0 & 0 & S_{36} \\ . & . & . & S_{44} & S_{45} & 0 \\ . & . & . & . & S_{55} & 0 \\ . & . & . & . & . & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} \tag{1}$$

Recently Zappalorto and Carraro [50] proposed to address the three-dimensional anisotropic problems with stress singularities by using the following displacement distributions:

$$u_x = f(z) \times u(x,y) \quad u_y = f(z) \times v(x,y) \quad u_z = g(z) \times w(x,y) \tag{2}$$

z being the thickness direction. One should note that the conventional plane strain case is fully included in this treatment by choosing a constant function for $g(z)$. In the same paper it was also proved that Eq. (2) allows to significantly simplify of the eigen-value problem by reducing the complete set of the elasticity equations governing the three-dimensional problem to the following system of two un-coupled equations [50]:

$$\begin{cases} B_{22}\frac{\partial^4\phi}{\partial x^4} - 2B_{26}\frac{\partial^4\phi}{\partial x^3\partial y} + (2B_{12}+B_{66})\frac{\partial^4\phi}{\partial x^2\partial y^2} - 2B_{16}\frac{\partial^4\phi}{\partial y^3\partial x} + B_{11}\frac{\partial^4\phi}{\partial y^4} = 0 \\ C_{55}\frac{\partial^2 w}{\partial x^2} + 2C_{45}\frac{\partial^2 w}{\partial x\partial y} + C_{44}\frac{\partial^2 w}{\partial y^2} = 0 \end{cases} \tag{3a, b}$$

where:

$$B_{ij} = S_{ij} - \frac{S_{i3}^2}{S_{33}} \text{ if } i=j \quad B_{ij} = S_{ij} - \frac{S_{i3}S_{j3}}{S_{33}} \text{ if } i \neq j \tag{3c}$$

C_{ij} and S_{ij} are stiffness and compliance coefficients, respectively, ϕ is the two-dimensional Airy stress function and w is defined in Eq. (2).

It is noteworthy that (3a) is the equation governing the solution of the corresponding anisotropic plane strain problem, whereas Eq. (3b) is the equation governing the antiplane problem of the anisotropic theory of elasticity. It is also worth mentioning that for transversally isotropic plates, as well as for isotropic plates, Eq. (3a,b) simplify as:

$$\begin{cases} \nabla^4\phi = 0 \\ \nabla^2 w = 0 \end{cases} \tag{3d, e}$$

matching the solution for isotropic materials by Zappalorto and Lazzarin [52,53] and being very similar to the solution proposed by Kotousov and Wang [54,55] for transversally isotropic plates.

3. In-plane stress field solution for anisotropic bimaterial corners

Consider a bi-material sharp wedge in an anisotropic plate (Fig. 1). Materials 1 and 2 are supposed to have a plane of symmetry which coincides with the plane of reference for the deformation field (rectilinear anisotropy), so that the in-plane and the antiplane problems are uncoupled by the material point of view (see Section 2).

For each of the considered sub-domains (materials) the following complex variables can be introduced:

$$z_1^{(k)} = x + \mu_1^{(k)} y = r\rho_1^{(k)} e^{i\theta_1^{(k)}} \quad z_2^{(k)} = x + \mu_2^{(k)} y = r\rho_2^{(k)} e^{i\theta_2^{(k)}} \tag{5}$$

where the superscript (k) allows to make distinction between the two materials ($k = 1, 2$) and:

$$\begin{aligned} \rho_j^{(k)} &= \sqrt{\left(\cos\theta + \alpha_j^{(k)}\sin\theta\right)^2 + \left(\beta_j^{(k)}\sin\theta\right)^2} \\ \theta_j^{(k)} &= \mathrm{Arg}\left(\cos\theta + \alpha_j^{(k)}\sin\theta + i\beta_j^{(k)}\sin\theta\right) \end{aligned} \tag{6}$$

$\mu_1^{(k)}$ and $\mu_2^{(k)}$ are unequal complex numbers defined as [51]:

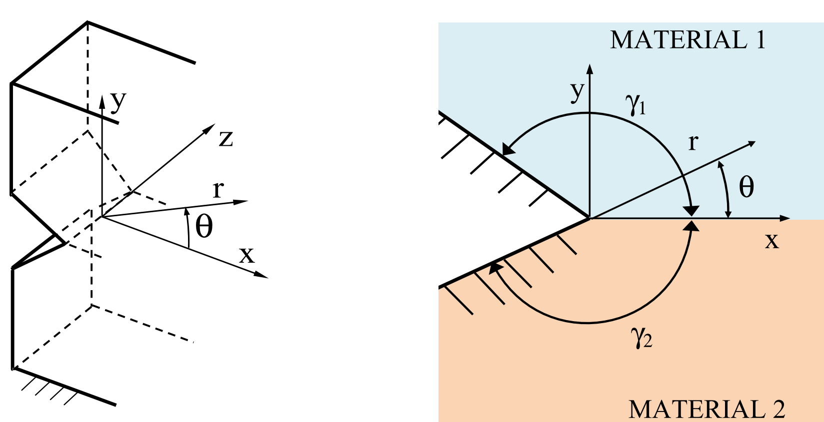

Fig. 1. Coordinates at a three dimensional V-notch tip.