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Formulation of enriched macro elements using trigonometric shear deformation theory for free vibration analysis of symmetric laminated composite plate assemblies

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ABSTRACT

The formulation of an enriched macro element suitable to analyze the free vibration response of composite plate assemblies is presented in this article. Based on the Trigonometric Shear Deformation Theory (TSDT) and the use of Gram–Schmidt orthogonal polynomials as enrichment functions a finite macro element is developed. In the TSDT framework, shear stresses are vanished at the top and bottom surfaces of the plates and shear correction factors are no longer required. The Principle of Virtual Work is applied to derive the governing equations of motion. A special connectivity matrix is obtained; so that hierarchically enriched global stiffness matrix and mass matrix of general laminated plate structures are derived, allowing to study generally coplanar plate assemblies by combining two or more macro elements. This procedure gives a matrix-eigenvalue problem that can be solved with optimum efficiency. Results of free vibration analysis for symmetric laminated plates of different thickness ratios, geometrical planform shapes and boundary conditions are presented. The accuracy of the formulation is ensured by comparing some numerical examples with those available in the literature.

1. Introduction

Currently, composite materials have become indispensable in several applications, such as high-performance structures in many fields of civil, marine and aerospace engineering, among others.

While composite materials have many advantages over conventional materials, they also present complex and challenging problems for structural analysis and design engineers and researchers. Many structural elements, such as cylinders, beams, plates and shells, can potentially be used for the analysis of composite laminates. Particularly, laminated plates of fiber reinforced composite materials, with different shapes, have great advantages and are widely used in high-performance structural components. During the last few decades the use of composite plates has increased in various engineering applications. The laminated plates are attractive structural components in many industries, because of the high stiffness-weight ratio along with the possibility to tailor the lamination scheme, which could be adapted to the requirements of design.

The global deformation of laminated composite plates is, in the general, characterized by complex couplings between extension, bending, torsion and shear modes. Furthermore, due to their low transverse shear stiffness, laminated composite plates exhibit a much more significant transverse shear deformation than homogeneous isotropic plates with the same geometric dimensions, even for low thickness-to-length ratios. In order to consider these aspects into the analysis and design of laminated plates, and to exploit the potential advantages of these materials, it is necessary to develop methodologies that include these effects. Furthermore, it is necessary to have accurate analysis tools that allow arrive to appropriate and versatile designs, according to increasingly stringent requirements.

For the study of plates with any thickness-to-length ratio, the formulations based on Equivalent Single Layer (ESL) theories must include the higher-order effects. This is accomplished by applying theories which take into account the shear deformation in the cinematic expressions, with the advantage of not requiring the use of shear correction factors and reproducing more accurately the distribution of interlaminar stresses in thick plates. Interesting *Higher*

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Order Shear Deformation Theories (HOSDT) were proposed by authors like Reddy and Liu [1], Touratier [2], Soldatos [3], Kant and Swaminathan [4], Karama et al. [5], among others, and more recently, by Mantari et al. [6], Mantari and Guedes Soares [7] and Grover et al. [8]. These theories satisfy the boundary conditions corresponding to the free surfaces of the plates and represent approximately the parabolic distribution of shear stress in the thickness. Most of the HOSD existing theories have polynomial expressions for the shear deformation. For example, in the general formulations presented by Carrera [9], Carrera et al. [10] and Demasi [11–15] the unknown variables are represented throughout the thickness by polynomial functions.

However, in relation to the ESL theories, in accordance with the reviews made by the authors and also with Mantari et al. [16], it is quite important to explore the behavior of other functions in the implementation of new shear deformation theories. It can be said that there is evidence of the demand generated by the higher order trigonometric theories [17], fundamentally due to the fact that they are much richer than polynomial functions, which are at the same time, simpler and more precise, and the boundaries conditions in the free surfaces of the plate are guaranteed at priori.

The above mentioned trigonometric theories were applied by Shimpi and Ainapure [18] and later by Arya et al. [19] for studying laminated beams. Subsequently, Ferreira et al. [20] used for the first time a trigonometric shear deformation theory for modeling symmetric laminated square plates by a meshless method, based on global multiquadric radial basis functions, obtaining very good results. Then, Roque et al. [21] used this trigonometric theory for laminated plates but incorporating the concept of multilayer laminates, obtaining very good results for the static analysis of symmetric laminated square plates. Starting from these studies, Xiang and Wang [22] presented the analysis of free vibrations of square laminated plates, using the trigonometric shear deformation theory and inverse multiquadric radial basis functions, arriving to very good results of natural vibration frequencies for different material and geometric parameters. Recently, Mantari et al. [17,23] presented a trigonometric shear deformation theory to model laminated composite and sandwich plates, with square or rectangular planform, by formulating a discrete finite element. Then, Mantari and Guedes Soares [7,24,25] completed these studies with the analysis of graded plates and advanced composite plates, respectively.

Regarding the analysis of thick plates of general geometries, Ramesh et al. [26] presented a higher-order triangular plate element based on the *Third-order Shear Deformation Theory* and a layer-wise plate theory of Reddy for the bending analysis of laminated composite plates. Zamani et al. [27] presented a transformation of coordinates combined with the differential equations from *First Order Shear Deformation Theory*, to model the problem of free vibration of laminated plates with trapezoidal and skew planform with different geometrical parameters, various aspect ratios and boundary conditions.

Using a *Higher Order Shear Deformation Theory*, Fazzolari et al. [28] present an exact dynamic stiffness method for free vibration analysis of composite plate assemblies. Previously, Houmat and Rashid [29] presented a method for coupling isoparametric cubic quadrilateral h-elements and straight sided serendipity quadrilateral p-elements for the free vibration analysis of plates with curvilinear planforms.

In previous papers, the authors have presented the formulation of hierarchical finite macro elements (*h-p* version of FEM), enriched with Gram–Schmidt orthogonal polynomials, using the *Classical Laminated Plates Theory* (CLPT) [30] and the *First Order Shear Deformation Theory* (FSDT) [31–33]. In this paper, the concept of macro element formulated by the authors is extended, so as to incorporate the kinematic corresponding to the *Trigonometric Shear Deformation Theory* (TSDT), which allows the study of thick laminated plates and plate assemblies, due the incorporation of mapping and assembly techniques.

2. Displacement field

A general quadrilateral thick laminated plate element, as shown in Fig. 1, is considered. A laminate of uninform thickness h with N_l layers is adopted for the analysis. Each layer consists of unidirectional fiber reinforced composite material. The fiber angle of kth layer counted from the surface $z = -h/2$ is β and it is measured from the x axis to the fiber direction. Symmetric lamination of plies is considered in this work (see Fig. 1A and C).

Based on the Trigonometric Shear Deformation Theory (TSDT) and taking into account the corresponding hypothesis [20], the displacement field can be described as:

$$\begin{aligned}\bar{u}(x,y,z,t) &= -z\frac{\partial w_0(x,y,t)}{\partial x} + \sin\frac{\pi z}{h}\phi_x(x,y,t)\\ \bar{v}(x,y,z,t) &= -z\frac{\partial w_0(x,y,t)}{\partial y} + \sin\frac{\pi z}{h}\phi_y(x,y,t)\\ \bar{w}(x,y,z,t) &= w_0(x,y,t)\end{aligned} \tag{1}$$

where $\bar{u},\ \bar{v},\ \bar{w} = w_0$ are the displacements of a generic point on the mid plane ($z = 0$) along (x, y, z) and (ϕ_x, ϕ_y) are the rotations of the transverse normal about y and x axis respectively. During free vibration, the displacements are assumed split in the spatial and temporal parts, being the last one periodic in time: $w_0(x,\ y,\ t) = w\ (x,\ y)\ \sin(\omega t)$ where ω is the radian natural frequency.

The linear strains associated with the displacement fields (Eq. (1)) are given by:

$$\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix} = \sin\frac{\pi z}{h}\begin{Bmatrix}\frac{\partial\phi_x}{\partial x}\\ \frac{\partial\phi_y}{\partial y}\\ \frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\end{Bmatrix} + z\begin{Bmatrix}-\frac{\partial^2 w}{\partial x^2}\\ -\frac{\partial^2 w}{\partial y^2}\\ -2\frac{\partial^2 w}{\partial x\partial y}\end{Bmatrix}$$
$$\begin{Bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{Bmatrix} = \frac{\pi}{h}\cos\frac{\pi z}{h}\begin{Bmatrix}\phi_y\\ \phi_x\end{Bmatrix} \tag{2}$$

3. Equations for dynamic analysis

The governing equations of the problem come from the dynamic version of the Principle of Virtual Work.

$$\int_{t_1}^{t_2}(\delta L)dt = 0 \tag{3}$$

where L is the lagrangian and is defined as $L = T - (U + V)$ where U, V, T are the strain energy, work done by applied forces and kinetic energy, respectively. In this article the term V is omitted as the analysis is limited to free vibration response.

The virtual strain energy δU is given by:

$$\delta U = \int_R\left\{\int_{-h/2}^{h/2}\left[\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + \tau_{xy}\delta\gamma_{xy} + \tau_{xz}\delta\gamma_{xz} + \tau_{yz}\delta\gamma_{yz}\right]dz\right\}dxdy \tag{4}$$

where R is the mid-surface area of the plate (see Fig. 1).

Replacing Eq. (2) into Eq. (4), the following expression is obtained:

$$\begin{aligned}\delta U = \int_R\Bigg\{\int_{-h/2}^{h/2}\Bigg[&\sigma_{xx}z\left(-\frac{\partial^2\delta w_0}{\partial x^2}\right) + \sigma_{xx}\frac{\partial\delta\phi_x}{\partial x}\sin\frac{\pi z}{h} + \sigma_{yy}z\left(-\frac{\partial^2\delta w_0}{\partial y^2}\right)\\ &+\sigma_{yy}\frac{\partial\delta\phi_y}{\partial y}\sin\frac{\pi z}{h} + \tau_{xy}z\left(-2\frac{\partial^2\delta w_0}{\partial x\partial y}\right) + \tau_{xy}\sin\frac{\pi z}{h}\left(\frac{\partial\delta\phi_x}{\partial y} + \frac{\partial\delta\phi_y}{\partial x}\right)\\ &+\tau_{xz}\delta\phi_x\frac{\pi}{h}\cos\frac{\pi z}{h} + \tau_{yz}\delta\phi_y\frac{\pi}{h}\cos\frac{\pi z}{h}\Bigg]dz\Bigg\}dxdy\end{aligned} \tag{5}$$