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# Elastoplastic buckling of axially loaded functionally graded material cylindrical shells

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ABSTRACT

In this paper, elastoplastic buckling behaviors of functionally graded material cylindrical shells under axial compression are investigated with Donnell shell theory and $J_2$ flow constitutive relation of functionally graded materials. The nonlinear material properties vary smoothly through the thickness, and a multi-linear hardening elastoplasticity is considered in the analysis. The buckling government equations are solved by Galerkin method, and the semi-analytical solution of the critical load is given. Numerical results from the present theory are derived by an iterative procedure. The theoretical elastoplastic critical loads are well verified by those of ABAQUS code, which includes both the material and geometrical nonlinearities. The elastic, elastoplastic, and plastic buckling regions of functionally graded cylindrical shells can be effectively distinguished through the present method, and various effects of the material nonlinearity, the dimensional parameters and the power law exponent are investigated.


## 1. Introduction

Functionally graded materials (FGMs) are new composites, made from a mixture of ceramic and metal constituents [1]. The mixture ratio of the two constituents changing smoothly, and the material properties vary continually through the thickness, which can greatly avoid stress concentration aroused by discontinuity of material properties, typically seen in laminate and fiber-reinforced composites.

In the research field of FGM plates and shells, many researches have been addressed for the elastic buckling problems. Javaheri and Eslami [2] and Malekzadeh et al. [3] investigated buckling of rectangular FGM plates under in-plane loads. Li and Batrab [4] and Wu et al. [5] focused on buckling and thermal buckling issues. Shen's works cover a wild range of postbuckling problems [6]. Meanwhile, Sofiyev [7] systemically study the linear dynamic buckling.

The elastoplastic or plastic buckling performances of plates and shells have been investigated extensively, and this research field is one of most important components in structural stability theory. Durban and Zuckerman [8], Kadkhodayan and Maarefdoust [9] concerned with elastoplastic buckling of a rectangular plate, with various boundary conditions, under uniform uniaxial or biaxial compression. Mao and Lu [10] investigated plastic buckling of homogeneous cylindrical shells under axial compression by both $J_2$ flow and deformation theories. Although many literatures had been addressed for the elastoplastic buckling behaviors of homogeneous plates and shells, little had been concerned with those of composite ones, especially for FGM plates and shells.

The continuously varying material properties of FGMs can be depicted by a homogenized mixture rule (named TTO model), initially proposed for metal alloys by Tamura et al. [11]. By introducing a proper stress transfer parameter, it can be used in FGMs [12,13]. An inverse analysis procedure, based on indentation tests, has been developed by Nakamura et al. [14,15] to identify constitutive parameters. With this model, Jahromi et al. [16] investigated residual stress during the fabrication of FGM vessels, and Jin et al. [17] studied the fracture issues in elastic–plastic FGMs. Besides, Akis [18] presented elastoplastic analysis for internally pressurized FGM spherical pressure vessels using small deformation theory. Results showed different modes of plasticization from the homogeneous spherical pressure vessel may take place due to the radial variation of the grading parameters. In this paper, elastoplastic buckling behaviors of FGM cylindrical shells under uniform axial compression are investigated by employing Donnell shell theory and $J_2$ flow constitutive relation of FGMs.

## 2. Formulation

For an uniform axial compressed FGM cylindrical shell, with thickness $h$, length $L$, and mean radius $R$, the coordinate system is placed on the middle surface of the shell, with the origin $o$ at its end and the coordinate axes $x$, $y$, and $z$ in the axial, circumferential, and the inward normal directions, as shown in Fig. 1.

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The incremental strain components of thin cylindrical shells are

$$\varepsilon_1 = \varepsilon_1^0 + zK_1 \tag{1}$$

where $\varepsilon_1 = [\varepsilon_{xx1} \;\; \varepsilon_{yy1} \;\; \varepsilon_{xy1}]^{\mathrm{T}}, K_1 = [K_{xx1} \;\; K_{yy1} \;\; K_{xy1}]^{\mathrm{T}}, \varepsilon_1^0 = [\varepsilon_{xx1}^0 \;\; \varepsilon_{yy1}^0 \;\; \varepsilon_{xy1}^0]^{\mathrm{T}}$. $\varepsilon_{xx}^0, \varepsilon_{yy}^0, \gamma_{xy}^0$ are the strain components on the middle surface and $K_{xx}$, $K_{yy}$, $K_{xy}$ are the curvature components and the subscript "1" denotes the increment of the corresponding parameters.

According to the nonlinear von Kárman strain–displacement relations, we have

$$\varepsilon_{xx1}^0 = u_{1,x} + \frac{1}{2}(w_{1,x})^2, \varepsilon_{yy1}^0 = v_{1,y} - \frac{w_1}{R} + \frac{1}{2}(w_{1,y})^2,$$
$$\varepsilon_{xy1}^0 = u_{1,y} + v_{1,x} + w_{1,x}w_{1,y} \tag{2}$$

$$K_{xx1} = -w_{1,xx}, \quad K_{yy1} = -w_{1,yy}, \quad K_{xy1} = -2w_{1,xy} \tag{3}$$

where $u$, $v$, $w$ are displacements along $x$, $y$, $z$, and the subscript comma denotes partial derivative, such as $u_{1,x} = \frac{\partial u_1}{\partial x}$, $w_{1,xy} = \frac{\partial^2 w_1}{\partial x \partial y}$.

Using the constitutive relation of FGM given in Section 3, the incremental stress components are given as

$$\sigma_1 = A\varepsilon_1 \tag{4}$$

where $\sigma_1 = [\sigma_{xx1} \quad \sigma_{yy1} \quad \sigma_{xy1}]^{\mathrm{T}}$ and the matrix $A$ is given in the Appendix A.

For thin cylindrical shells, the incremental internal force and moment components

$$\{N_{ij1}, M_{ij1}\} = \int_l \sigma_{ij1}\{1, z\}\mathrm{d}z, \quad (i,j = x,y) \tag{5}$$

It can be rewritten in matrix form as

$$\begin{bmatrix} N_1 \\ M_1 \end{bmatrix} = \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{B} & \widetilde{D} \end{bmatrix} \begin{bmatrix} \varepsilon_1^0 \\ K_1 \end{bmatrix} \tag{6}$$

in which $N = [N_{xx} \quad N_{yy} \quad N_{xy}]^{\mathrm{T}}$, $M = [M_{xx} \quad M_{yy} \quad M_{xy}]^{\mathrm{T}}$ and $\widetilde{A}, \widetilde{B}, \widetilde{D}$ are defined in the Appendix A. It should be noted that, the integral range $l$ should be divided into two subsections according to the material stress state.

The above equation can be rewritten as

$$\begin{bmatrix} \varepsilon_1^0 \\ M_1 \end{bmatrix} = \begin{bmatrix} \widehat{A} & \widehat{B} \\ \widehat{C} & \widehat{D} \end{bmatrix} \begin{bmatrix} N_1 \\ K_1 \end{bmatrix} \tag{7}$$

where $\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}$ are given in Appendix A.

The basic equilibrium equations of cylindrical shells of Donnell type are

$$N_{xx,x} + N_{xy,y} = 0, \quad N_{xy,x} + N_{yy,y} = 0$$
$$M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} + \frac{N_{yy}}{R} + N_{xx}w_{,xx} + 2N_{xy}w_{,xy} + N_{yy}w_{,yy} + q = 0 \tag{8}$$

in which, $q$ is the lateral pressure.

The deflection, internal force and moment of the shell can be divided as

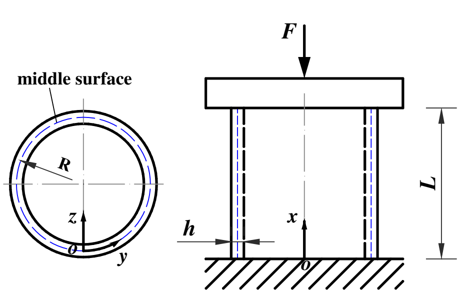


**Fig. 1.** Axial compressed FGM cylindrical shell.

$$w = w_0 + w_1, \quad N_{ij} = N_{ij0} + N_{ij1}, \quad M_{ij} = M_{ij0} + M_{ij1} \tag{9}$$

where the subscript "0" denotes the prebuckling state. Then the incremental form of the above equations are given as

$$N_{xx1,x} + N_{xy1,y} = 0, \quad N_{xy1,x} + N_{yy1,y} = 0$$
$$M_{xx1,xx} + 2M_{xy1,xy} + M_{yy1,yy} + \frac{N_{yy1}}{R} + N_{xx0}w_{1,xx} + 2N_{xy0}w_{1,xy} + N_{yy0}w_{1,yy} = 0 \tag{10}$$

The deformation compatible equation of cylindrical shell is given by Eq. (2).

$$\varepsilon_{xx1,yy}^0 + \varepsilon_{yy1,xx}^0 - \varepsilon_{xy1,xy}^0 = -\frac{w_{1,xx}}{R} + (w_{1,xy})^2 + w_{1,xx}w_{1,yy} \tag{11}$$

Introducing the Airy's stress function $\varphi_1(x,y)$ which satisfied

$$N_{xx1} = \varphi_{1,yy}, \quad N_{yy1} = \varphi_{1,xx}, \quad N_{xy1} = -\varphi_{1,xy} \tag{12}$$

Thus, the first two equations satisfied automatically.

By eliminating the nonlinear terms, and substituting Eqs. (12) into Eqs. (7), then, Eq. (11) and the last equation of Eq. (10) turns into

$$\mathrm{I}_1^{\varphi_1}\varphi_1 + \mathrm{I}_1^{w_1}w_1 + \frac{1}{R}w_{1,xx} = 0$$
$$\mathrm{I}_2^{\varphi_1}\varphi_1 + \mathrm{I}_2^{w_1}w_1 + \frac{1}{R}\varphi_{1,xx} + N_{xx0}w_{1,xx} + 2N_{xy0}w_{1,xy} + N_{yy0}w_{1,yy} = 0 \tag{13}$$

where $\mathrm{I}_i^p$ $(i = 1,2,\; p = \varphi_1, w_1)$ are differential operators defined as follows.

$$\mathrm{I}_i^p = \Gamma_{5i-4}^p p_{,yyyy} + \Gamma_{5i-3}^p p_{,xyyy} + \Gamma_{5i-2}^p p_{,xxyy} + \Gamma_{5i-1}^p p_{,xxxy} + \Gamma_{5i}^p p_{,xxxx} \tag{14}$$

in which $\Gamma_{5i-j}^p$ $(j = 1,2,3,4)$ are given in Appendix A.

Eqs. (13) represent the buckling government equations of elastoplastic FGM cylindrical shells, which will be used to derive the buckling critical condition.

## 3. Material constitutive relation

FGMs are inhomogeneous materials of smoothly varying ceramic/metallic mixture ratio through the thickness. Generally, the volume fraction of the ceramic constitute is assumed to submit the power law distribution as [6]

$$V_{\mathrm{c}} = (0.5 + z/h)^k, \quad V_{\mathrm{c}} + V_{\mathrm{m}} = 1 \tag{15}$$

where $V$ denotes the volume fraction of constituents. The subscripts c, m respectively correspond to the ceramic and metallic constitutes. $k$ is the power law exponent, which is a critical parameter to control the distribution of the constituents in FGMs.

In general, ceramic materials are brittle materials of relatively higher elastic modulus and strength than those of metallic materials, which are typically ductile materials. According to the TTO model for FGMs, the ceramic constituents in FGMs are assumed to be elastic when deformation takes places. Material flows arouse mainly by the plastic flowing of the metallic constituent when the stress state beyond its yield limit. Thus, the multi-linear hardening elastoplastic material properties of FGMs along the thickness can be defined as [14,17].

$$E = \left(\frac{q + E_{\mathrm{c}}}{q + E_{\mathrm{m}}}E_{\mathrm{m}}V_{\mathrm{m}} + E_{\mathrm{c}}V_{\mathrm{c}}\right) \Big/ \left(\frac{q + E_{\mathrm{c}}}{q + E_{\mathrm{m}}}V_{\mathrm{m}} + V_{\mathrm{c}}\right)$$
$$v = v_{\mathrm{m}}V_{\mathrm{m}} + v_{\mathrm{c}}V_{\mathrm{c}}$$
$$\sigma_{\mathrm{Y}} = \sigma_{\mathrm{Ym}}\left(V_{\mathrm{m}} + \frac{q + E_{\mathrm{m}}}{q + E_{\mathrm{c}}}\frac{E_{\mathrm{c}}}{E_{\mathrm{m}}}V_{\mathrm{c}}\right) \tag{16}$$
$$H = \left(\frac{q + E_{\mathrm{c}}}{q + H_{\mathrm{m}}}H_{\mathrm{m}}V_{\mathrm{m}} + E_{\mathrm{c}}V_{\mathrm{c}}\right) \Big/ \left(\frac{q + E_{\mathrm{c}}}{q + H_{\mathrm{m}}}V_{\mathrm{m}} + V_{\mathrm{c}}\right)$$