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Stress fields at sharp angular corners in thick anisotropic composite plates

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ABSTRACT

A new three-dimensional theory to be applied to thick anisotropic plates is developed, according to which the 3D governing equations of elasticity are successfully reduced to the simultaneous solution of two uncoupled equations in the two-dimensional space. With the new theory an analytical solution for the three-dimensional stress fields in anisotropic composite plates with V-notches is presented and its degree of accuracy is discussed comparing theoretical results and numerical data from 3D FE analyses.

1. Introduction

The knowledge of the stress fields near the tip of defects like cracks, inclusions or sharp notches is essential in the design of mechanical components. Even if in the past and recent literature many efforts have been devoted to study notch stress distributions in isotropic media, the stress analysis of anisotropic plates is important as well, being them interesting for many engineering applications such as composites, crystals, wood and reinforced polymers.

Independently of the far applied loads, the stress state close to a geometrical variation, such as a hole or a notch, is inherently multiaxial; under such a stress state the fatigue behaviour of composite materials might be very complex as highlighted by a number of publications [1–7]. It is also worth mentioning, the accurate knowledge of local stress fields is essential for formulating strength assessment rules of structural components, engineering strength criteria being almost based on quantities which can be directly linked to the stress distributions [8,9].

The stress state close to a crack in a two-dimensional anisotropic plate was analysed many years ago by Sih et al. [10], who showed that the elastic stress and strain singularity remains 0.5, as for the isotropic case and pointed out that the near tip fields can be written in terms of three real stress intensity factors. Different from the crack case, the anisotropic singularity degree of re-entrant corners depends both on the notch opening angle, 2α, and on the material elastic properties [11–18].

Recently, plane stress fields around square and rectangular holes in symmetric laminates have been derived by Rao et al. [19] using Savin's basic solution for anisotropic plates and generalised hole mapping functions, while Ukadgaonker and Kakhandki [20] carried out an analytical study of the stress distributions in an orthotropic plate with an irregular shaped hole and different in-plane loading conditions.

All the above mentioned works dealt with a two-dimensional analysis of the problem, whilst only few works have considered the importance of the three-dimensional nature of the stress fields in cracked or notched composite plates.

Among these, Choi and Folias [21] studied the three-dimensional stress fields in a laminated composite plate weakened by a hole and subjected to a uniform displacement along the horizontal direction.

By adopting the Kane and Mindlin's assumption, Kotousov and Wang developed a generalised plane-strain theory for transversely isotropic composite plates and determined closed form solutions for the three-dimensional stresses, especially the through-the-thickness component, around a circular hole and a circular inclusion in thick plates [22,23].

A FE investigation on the three-dimensional nature of stress fields near the tip region of a cracked orthotropic plate has been carried out by Prabhu and Lambros [24], who also discussed the relative extent of regions of three-dimensional to two-dimensional deformation in the cracked plate.

An eigenfunction expansion technique was used by Chaudhuri [25] to derive the three-dimensional asymptotic stress field in the vicinity of the front of a semi-infinite through-the-thickness crack weakening an infinite transversely isotropic unidirectional fibre reinforced composite plate. The same approach was later extended

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by Chaudhuri and Yoon [26,27] to study three-dimensional asymptotic displacement and stress fields in three-material plates.

In this paper we address the problem of the three-dimensional stress fields near pointed V-notches in anisotropic plates of finite thickness. By assuming separation of variables for the displacement functions of the three-dimensional problem, the 3D governing equations of anisotropic elasticity are reduced to two uncoupled differential equations; one is related to the solution of the corresponding plane strain notch problem, the other one, instead, provides the solution of the corresponding out-of-plane shear notch problem.

The explicit formulae for the in-plane and the out-of-plane shear stress fields at the vertex of sharp V-notches in anisotropic plates are later derived and given in closed form as a function of generalised stress intensity factors. The accuracy of the analytical expressions is checked versus a number of finite element analyses carried out on thick composite plates with rectangular cutouts or pointed V notches, documenting a very satisfactory agreement.

2. The three-dimensional anisotropic elasticity problem

In this work a rectilinear anisotropic material is considered, according to which the in-plane and the antiplane problems are, by the material point of view, uncoupled. Conventional fibre-reinforced polymer composites respect this hypothesis. Under this circumstance, stress–strain relationship can be written as:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ \cdot & S_{22} & S_{23} & 0 & 0 & S_{26} \\ \cdot & \cdot & S_{33} & 0 & 0 & S_{36} \\ \cdot & \cdot & \cdot & S_{44} & S_{45} & 0 \\ \cdot & \cdot & \cdot & \cdot & S_{55} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} \tag{1}$$

Consider now a pointed V-notch in a thick plate, with a Cartesian coordinate system centred at the notch tip (see Fig. 1), and suppose that displacement distributions close to the apex can be written, through separation of variables, in the following form:

$$u_x = f(z) \times u(x,y) \quad u_y = f(z) \times v(x,y) \quad u_z = g(z) \times w(x,y) \tag{2}$$

where $f(z)$ and $g(z)$ can be regarded as generic polynomial functions of arbitrary order. This represents a refinement of the proposal in Refs. [28–30].

Thanks to Eq. (2), strain components result:

$$\begin{aligned} &\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = f(z) \times \frac{\partial u(x,y)}{\partial x} \quad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = f(z) \times \left(\frac{\partial u(x,y)}{\partial y} + \frac{\partial v(x,y)}{\partial x} \right) \\ &\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = f(z) \times \frac{\partial v(x,y)}{\partial y} \quad \gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = f'(z) \times u(x,y) + g(z) \times \frac{\partial w(x,y)}{\partial x} \\ &\varepsilon_{zz} = \frac{\partial u_z}{\partial z} = g'(z) \times w(x,y) \quad \gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} = f'(z) \times v(x,y) + g(z) \times \frac{\partial w(x,y)}{\partial y} \end{aligned} \tag{3}$$

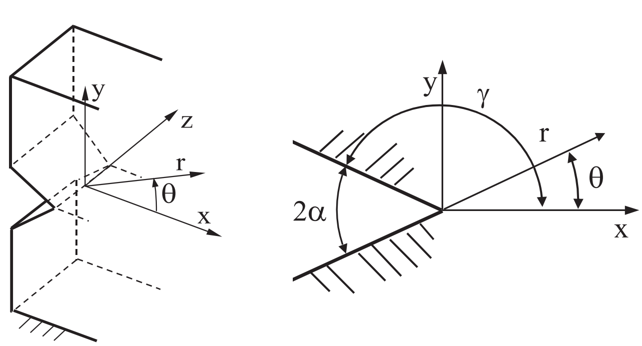

Fig. 1. Coordinates at a three dimensional V-notch tip.

Eq. (3) can be further simplified by noting that, due to the geometrical singularity at the V-notch vertex, singular strains are expected when approaching the notch tip (as $r = \sqrt{x^2 + y^2}$ tends toward zero). Displacement functions u, v and w must be finite, instead. Accordingly:

$$\begin{aligned} \gamma_{xz} &\cong g(z) \times \frac{\partial w(x,y)}{\partial x} \\ \gamma_{yz} &\cong g(z) \times \frac{\partial w(x,y)}{\partial y} \end{aligned} \tag{4}$$

and ε_{zz} can be regarded as negligible, so that a plane strain condition is approximately present at the notch tip:

$$\begin{aligned} \sigma_{zz} &\cong -\frac{S_{13}\sigma_{xx} + S_{23}\sigma_{yy} + S_{36}\tau_{xy}}{S_{33}} \\ &\cong f(z)\left(C_{13}\frac{\partial u}{\partial x} + C_{23}\frac{\partial v}{\partial y} + C_{36}\frac{\partial u}{\partial y} + C_{36}\frac{\partial v}{\partial x} \right) \end{aligned} \tag{5}$$

Moreover, Eqs. (3)–(5) guarantee stresses to be in the form:

$$\begin{aligned} \sigma_{xx} &= f(z) \times \tilde{\sigma}_{xx}(x,y) \\ \sigma_{yy} &= f(z) \times \tilde{\sigma}_{yy}(x,y) \\ \sigma_{zz} &= f(z) \times \tilde{\sigma}_{zz}(x,y) \\ \sigma_{xy} &= f(z) \times \tilde{\tau}_{xy}(x,y) \end{aligned} \tag{6}$$

$$\begin{aligned} \sigma_{xz} &= C_{55}\gamma_{xz} + C_{45}\gamma_{yz} = g(z)\left[C_{55}\frac{\partial w(x,y)}{\partial x} + C_{45}\frac{\partial w(x,y)}{\partial y} \right] = g(z) \times \tilde{\sigma}_{xz}(x,y) \\ \sigma_{yz} &= C_{45}\gamma_{xz} + C_{44}\gamma_{yz} = g(z)\left[C_{45}\frac{\partial w(x,y)}{\partial x} + C_{44}\frac{\partial w(x,y)}{\partial y} \right] = g(z) \times \tilde{\sigma}_{yz}(x,y) \end{aligned} \tag{7}$$

where C_{ij} are stiffness coefficients. Consider the following equilibrium equation in the z direction:

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \tag{8}$$

and substitute Eqs. (5) and (7):

$$\begin{aligned} &\frac{\partial}{\partial x}\left[g(z)C_{55}\frac{\partial w(x,y)}{\partial x} + g(z)C_{45}\frac{\partial w(x,y)}{\partial y} + f'(z)(C_{13}u + C_{36}v) \right] \\ &+ \frac{\partial}{\partial y}\left[g(z)C_{45}\frac{\partial w(x,y)}{\partial x} + g(z)C_{44}\frac{\partial w(x,y)}{\partial y} + f'(z)(C_{23}v + C_{36}u) \right] = 0 \end{aligned} \tag{9}$$

In this last equation the terms proportional to u and v can be neglected when compared to $\frac{\partial w}{\partial x}$ and to $\frac{\partial w}{\partial y}$, so that, finally:

$$C_{55}\frac{\partial^2 w}{\partial x^2} + 2C_{45}\frac{\partial^2 w}{\partial x \partial y} + C_{44}\frac{\partial^2 w}{\partial y^2} = 0 \tag{10}$$

Consider now equilibrium equations in the x- and in the y- directions. Thanks to Eqs. (6) and (7) we have:

$$\begin{aligned} &f(z)\left[\frac{\partial \tilde{\sigma}_{xx}}{\partial x} + \frac{\partial \tilde{\tau}_{xy}}{\partial y} \right] + g'(z)\left[C_{44}\frac{\partial w(x,y)}{\partial x} + C_{45}\frac{\partial w(x,y)}{\partial y} \right] = 0 \\ &f(z)\left[\frac{\partial \tilde{\tau}_{xy}}{\partial x} + \frac{\partial \tilde{\sigma}_{yy}}{\partial y} \right] + g'(z)\left[C_{45}\frac{\partial w(x,y)}{\partial x} + C_{55}\frac{\partial w(x,y)}{\partial y} \right] = 0 \end{aligned} \tag{12}$$

Differentiating the first equation by x and the second one by y and making summation, using the *Schwarz* theorem for partial derivatives and considering also Eq. (10), the result is:

$$\frac{\partial^2 \tilde{\sigma}_{xx}}{\partial x^2} + \frac{\partial^2 \tilde{\sigma}_{yy}}{\partial y^2} + 2\frac{\partial^2 \tilde{\sigma}_{xy}}{\partial y \partial x} = 0 \tag{13}$$

It is worth noting that Eq. (13) is inherently satisfied introducing the Airy (bi-dimensional) stress function $\phi(x,y)$ such that: