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An analytical solution for multilayered beams subjected to ends loads

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ABSTRACT

An alternative model for multilayered beams undergoing axial, shear and bending loads applied at the beam's ends is developed. It is based on a layer-wise kinematics, which inherently fulfills the equilibrium equations at layer level and the interface continuity conditions. This kinematics is suitably expressed by introducing a set of generalized variables representative of the beam midline displacement field, which become the primary variables of the problem governing equations. As a consequence, the proposed beam model exhibits the computational characteristics of an equivalent single layer model and possesses the accuracy of layer-wise beam theories, as well. Closed form solutions for different beam support and load conditions are given. Validation results are presented for composite laminates and functionally graded beams are investigated to show the potentiality of the presented beam theory.

1. Introduction

The proper modeling of composite multilayered configurations is crucial in composite structures design, because of the need for an accurate appraisal of their structural behavior responsible of complex damage and failure mechanisms. It is understood that 3-D analytical or numerical solutions represent the best choice in terms of solution accuracy. Examples are given, among others, by the works of Pagano [1–3], where the exact solutions for the bending problem of simply supported composite laminates under distributed normal surface forces are presented. A review on such topic is given by Bogdanovich and Yushanov, who also presented results obtained by a 3-D displacement-assumed variational analysis based on Bernstein polynomials [4]. However, exact 3-D solution are generally difficult to obtain and they usually refer to simple specific geometric and/or load and support configurations [5]. These drawbacks can be overcome by 3-D numerical approaches, such as finite elements [6–8] or boundary elements [9–11] whose application generally involves meaningful computational costs. Thus, the use of 1-D beam or 2-D plate formulations allows to reduce the computational costs while ensuring an appropriate level of accuracy [12] and, for such reason, they are widely employed particularly during the design phase.

1-D or 2-D laminated structures modeling approaches can be classified into layer-wise (LW) and equivalent single layer (ESL) theories [13–15]. The LW approach assumes through-the-thickness approximation of the displacements at layer level and consequently the displacement and stress continuity at interfaces needs to be explicitly enforced to recover the model of the laminate as a whole. On the other hand, in the classical ESL theories the kinematical model is assumed unique for the whole laminate, generally in the form of a through-the-thickness expansion of continuous functions. According with these modeling assumptions, LW theories result more accurate than the ESL ones in predicting the mechanical fields for moderately to very thick laminates. However, LW theories exhibit an increasing modeling cost as the number of laminate plies increases, since the number of involved degrees of freedom is strictly related to the number of layers. On the contrary, the computational effort associated with ESL models is independent on the number of layers although the accuracy loss as the laminate thickness increases. In addition, since the classical ESL approaches are characterized by a unique displacement assumption for the whole laminate, the through-the-thickness strain distributions result continuous crossing the interfaces and this implies that they are not capable of ensuring the equilibrium conditions at interfaces and thus are not able to proper model the interlaminar stress component distribution. Different approaches have been proposed in literature to improve basic ESL models. Some of these are the so called refined high order deformation theories that employ Taylor, trigonometric or exponential expansion, or even combination of them, to approximate the trough-the-thickness displacement distributions and improve the accuracy for moderately thick and thick laminate [12,16]. However, these kind of refined higher order shear deformation theories,

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as well as CLT and FSDT models, are not capable of fulfilling the so called C_z^0 requirements, which call for discontinuous strain and continuous transverse stress components crossing the layers' interface [5]. To cope with these requirements allowing at the same time to preserve the effectiveness of the ESL models, the so-called zig-zag theories have been proposed in the literature [17–19]. In these theories, a LW discontinuous function is a priori selected to enrich the kinematical model in such a way that the interface conditions, in terms of continuity of displacements and equilibrium of tractions, are met [20].

The present paper collocates in the framework of refined 1-D theories for multilayered composite beams. As far as a lot of works have been carried out on LW and ESL theories for 1-D multilayered structures, a comprehensive review of the proposed beam theories is out of the scope of the present paper and the interested reader is referred to Ref. [16] for the state of the art on the subject. Here, a new approach for the analysis of multilayered beam-type laminates undergoing end loads is proposed. It is based on a layer-wise kinematics, which is explicitly derived so as to fulfil the point-wise equilibrium balance equations as well as the traction-free conditions at the laminate top and bottom surfaces. In turn, generalized kinematical variables are introduced and the interface continuity conditions are suitably employed allowing for a reduction of the degrees of freedom to those needed for a single layer beam. As a consequence, the obtained model for multilayered beam presents the effectiveness of the ESL approaches preserving the accuracy of the LW models. It is also worth noting that in the proposed model the interlaminar stress continuity conditions are inherently fulfilled without the use of additionally independent stress/strain approximation functions. The solution for different beam configurations is presented and the model validated with results obtained by finite elements analyses. Finally, to evidence the outperforming computational characteristics of the proposed beam theory, functionally graded beam are analyzed by employing a discrete layer approach.

2. Basic relationships and assumptions

Let us consider a linearly elastic multilayered beam consisting of n orthotropic plies. The beam has length L and rectangular cross section of height h and unitary width. The top and bottom surfaces of the beam are traction free whereas generalized stress resultants are applied on the end sections. A global reference system $\{x, z\}$ is introduced with the origin centered at the left end of the beam midline and the x-axis coincident with the beam midline. The kth layer of thickness h_k is also referred to a local coordinate system denoted by $\{x, z_k\}$, which is centered at the kth layer midline, see Fig. 1. The distance $\bar{h}_k$ of the kth layer midline from the beam midline is given by

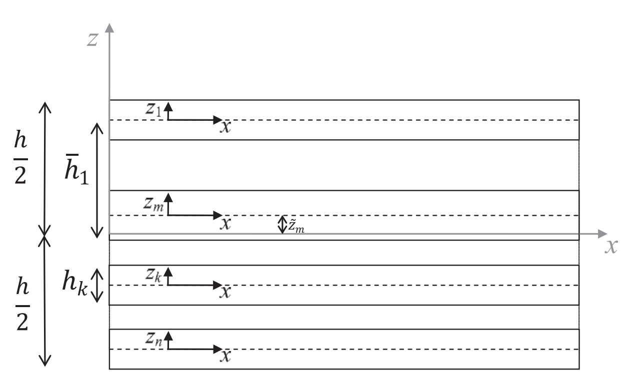

Fig. 1. Laminate geometrical scheme.

$$\bar{h}_k = \frac{h - h_k}{2} - \sum_{i=1}^{k-1} h_i \tag{1}$$

According to $\{3, 2\}$-order kinematic description [21], the displacement field inside each layer is assumed as

$$u^{\langle k\rangle}(x, z_k) = u_0^{\langle k\rangle}(x) + z_k \theta^{\langle k\rangle}(x) + z_k^2 B^{\langle k\rangle}(x) + z_k^3 C^{\langle k\rangle}(x) \tag{2a}$$

$$w^{\langle k\rangle}(x, z_k) = w_0^{\langle k\rangle}(x) + z_k A^{\langle k\rangle}(x) + z_k^2 D^{\langle k\rangle}(x) \tag{2b}$$

where $u_0^{\langle k\rangle}$ and $w_0^{\langle k\rangle}$ are the axial and transverse displacement components of the layer midline, $\theta^{\langle k\rangle}$ is the layer cross-sectional rotation whereas $A^{\langle k\rangle}$, $B^{\langle k\rangle}$, $C^{\langle k\rangle}$, and $D^{\langle k\rangle}$ are unknown functions of x to be determined. The superscript $\langle k\rangle$ is used to denote quantities associated with the k-th layer of the beam, whereas symbols not affected by this notation refer to the plate as a whole.

Assuming linear strain–displacement relationships, the strains components for the assumed kinematical model read as

$$\varepsilon_{xx}^{\langle k\rangle} = \frac{du_0^{\langle k\rangle}}{dx} + z_k \frac{d\theta^{\langle k\rangle}}{dx} + z_k^2 \frac{dB^{\langle k\rangle}}{dx} + z_k^3 \frac{dC^{\langle k\rangle}}{dx} \tag{3a}$$

$$\varepsilon_{zz}^{\langle k\rangle} = \frac{dA^{\langle k\rangle}}{dx} + 2z_k \frac{dD^{\langle k\rangle}}{dx} \tag{3b}$$

$$\gamma_{xz}^{\langle k\rangle} = \theta^{\langle k\rangle} + \frac{dw_0^{\langle k\rangle}}{dx} + z_k \left(\frac{dA^{\langle k\rangle}}{dx} + 2B^{\langle k\rangle} \right) + z_k^2 \left(\frac{dD^{\langle k\rangle}}{dx} + 3C^{\langle k\rangle} \right) \tag{3c}$$

The in-plane and transverse normal stresses σ_{xx} and σ_{zz} and the shear stress τ_{zx} are evaluated at the ply level using constitutive relationships, which write as

$$\sigma_{xx}^{\langle k\rangle} = Q_{xx}^{\langle k\rangle} \varepsilon_{xx}^{\langle k\rangle} + Q_{xz}^{\langle k\rangle} \varepsilon_{zz}^{\langle k\rangle} \tag{4a}$$

$$\sigma_{zz} = Q_{zx}^{\langle k\rangle} \varepsilon_{xx}^{\langle k\rangle} + Q_{zz}^{\langle k\rangle} \varepsilon_{zz}^{\langle k\rangle} \tag{4b}$$

$$\tau_{xz}^{\langle k\rangle} = G_{xz}^{\langle k\rangle} \gamma_{xz} \tag{4c}$$

where $Q_{ij}^{\langle k\rangle}$ are the layer stiffness coefficients and $G_{xz}^{\langle k\rangle}$ is the shear modulus.

By substituting Eq. (4) along with Eq. (3) into the stress equilibrium equations, the following set of equations is inferred

$$Q_{xx}^{\langle k\rangle} \frac{d^2 u_0^{\langle k\rangle}}{dx^2} + \left(Q_{xz}^{\langle k\rangle} + G_{xz}^{\langle k\rangle} \right) \frac{dA^{\langle k\rangle}}{dx} + 2G_{xz}^{\langle k\rangle} B^{\langle k\rangle} = 0 \tag{5a}$$

$$Q_{xx}^{\langle k\rangle} \frac{d^2 \theta^{\langle k\rangle}}{dx^2} + 6G_{xz}^{\langle k\rangle} C^{\langle k\rangle} + 2\left(Q_{xz}^{\langle k\rangle} + G_{xz}^{\langle k\rangle} \right) \frac{dD^{\langle k\rangle}}{dx} = 0 \tag{5b}$$

$$\left(Q_{xz}^{\langle k\rangle} + G_{xz}^{\langle k\rangle} \right) \frac{d\theta^{\langle k\rangle}}{dx} + G_{xz}^{\langle k\rangle} \frac{d^2 w_0^{\langle k\rangle}}{dx^2} + 2Q_{zz}^{\langle k\rangle} D^{\langle k\rangle} = 0 \tag{5c}$$

$$G_{xz}^{\langle k\rangle} \frac{d^2 A^{\langle k\rangle}}{dx^2} + 2\left(Q_{xz}^{\langle k\rangle} + G_{xz}^{\langle k\rangle} \right) \frac{dB^{\langle k\rangle}}{dx} = 0 \tag{5d}$$

$$3\left(Q_{xz}^{\langle k\rangle} + G_{xz}^{\langle k\rangle} \right) \frac{dC^{\langle k\rangle}}{dx} + G_{xz}^{\langle k\rangle} \frac{d^2 D^{\langle k\rangle}}{dx^2} = 0 \tag{5e}$$

$$\frac{d^2 B^{\langle k\rangle}}{dx^2} = \frac{d^2 C^{\langle k\rangle}}{dx^2} = 0 \tag{5f}$$

The previous equations hold for each kth single layer of the laminate. To ensure laminate interface continuity, they have to be supplemented by the compatibility of displacements and