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Accuracy of multiscale asymptotic expansion method

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ABSTRACT

The multiscale asymptotic expansion method (MsAEM) is generally implemented by finite element method. The calculating accuracy of MsAEM depends completely on the order of asymptotic expansion and the order of finite element. First, the necessary number of expansion term is decided in a mechanical view from the pseudo loads used for solving influence functions. Next for different order of load cases, the analytical solutions of the static problems of the periodical composite rod are obtained using different order of MsAEM and finite elements. In those solutions, the element order for solving analytical macro displacements depends on the external loads whereas the element orders for solving analytical influence functions are determined from the governing differential equations of influence functions. Then, two dimensional (2D) periodical composite are explored similarly. Finally, the potential energy functional is used to evaluate the accuracy of MsAEM, and numerical comparisons validate the conclusions.

1. Introduction

The composite materials and structures have been widely used in aerospace, automotive and marine engineering due to their high stiffness weight ratio. It is well known that the macro solutions, for example the lower order frequencies and mode shapes, can be solved efficiently for many composites to the satisfactory accuracy using the iso-strain or iso-stress model [1] and other homogenized approaches [2]. But the micro stress analysis is very expensive comparing with the macro analysis. To balance the accuracy and efficiency, various multiscale methods have been motivated, such as the mathematical homogenization method (MHM) [3,4], the generalized finite element method (GFEM) [5,6], the multiscale finite element (MsFEM) [7,8], the heterogeneous multiscale method (HMM) [9,10] and the multiscale eigenelement method (MEM) [11,12], among of which MHM is essential and representative and has been elaborated in many Refs. [13–21, for examples]. And the multiscale asymptotic expansion method (MsAEM) is widely used one of the mathematical homogenization methods. So far no paper investigates the effects of the expansion order and the element order used in calculation on the accuracy of MsAEM, or on the accuracy of influence function and the derivatives of macro displacements.

In this context, the objective of present study is to give the general principle to determine the necessary order of asymptotic expansion and the order of finite element in application, and the potential energy functional is used to evaluate the calculating accuracy of MsAEM. The outline of present paper is as follows: for two dimensional (2D) elastic composite, the governing differential equations for solving different order of influence functions are listed in Section 2, from which the necessary order of expansion term is derived; and in Section 3 the static problems of the periodical composite rod are studied through MsAEM with different order of expansion and finite elements; the 2D periodical composite are analyzed in Section 4. Finally, conclusions are drawn in Section 5.

2. The uncoupled multiscale asymptotic expansion method

Based on the assumptions of microstructure periodicity and uniformity of a unit cell domain, the homogenization theory decomposes the heterogeneous boundary value problem into the unit cell (micro) problem and the global (macro) problem, that means micro and macro problems can be solved independently. The cell problem must be solved prior to global problem for which the homogenized elastic constants are the input parameters.

First, a necessary description of the differential equations of MsAEM is presented below. This lays the foundation of present study, and the introduction focus on the strong form other than weak form. For simplicity, 2D periodical composite of two scales is taken into account in following introduction.

The governing equation for 2D composite is elliptic for most cases with multiscale or rough coefficients as

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Table 1
Perturbed displacements and governing equations of influence functions.

Order	Perturbed displacements	Governing equations of influence functions
1	$u_m^1(\boldsymbol{x},\boldsymbol{y}) = -\chi_{1m}^{kl}(\boldsymbol{y})\frac{\partial u_k^0(\boldsymbol{x})}{\partial x_l}$	$\nabla_{y_j}\cdot\left(E_{ijmn}^{\varepsilon}\nabla_{y_n}\chi_{1m}^{kl}\right) = \nabla_{y_j}\cdot E_{ijmn}^{\varepsilon}$
2	$u_m^2(\boldsymbol{x},\boldsymbol{y}) = -\chi_{2m}^{klp}(\boldsymbol{y})\frac{\partial^2 u_k^0(\boldsymbol{x})}{\partial x_l\partial x_p}$	$\nabla_{y_j}\cdot E_{ijmn}^{\varepsilon}\nabla_{y_n}\chi_{2m}^{klp} = -\nabla_{y_j}\cdot E_{ijmp}^{\varepsilon}\chi_{1m}^{kl} -E_{ipmn}^{\varepsilon}\nabla_{y_n}\chi_{1m}^{kl} + E_{ipkl}^{\varepsilon} - E_{ipkl}^{H}$
3	$u_m^3(\boldsymbol{x},\boldsymbol{y}) = -\chi_{3m}^{klpq}(\boldsymbol{y})\frac{\partial^3 u_k^0(\boldsymbol{x})}{\partial x_l\partial x_p\partial x_q}$	$\nabla_{y_j}\cdot E_{ijmn}^{\varepsilon}\nabla_{y_n}\chi_{3m}^{klpq} = -E_{iqmn}^{\varepsilon}\nabla_{y_n}\chi_{2m}^{klp} -\nabla_{y_j}\cdot E_{ijmq}^{\varepsilon}\chi_{2m}^{klp} - E_{iqmp}^{\varepsilon}\chi_{1m}^{kl}$

Table 2
Perturbed displacements and governing equations of influence functions for a periodical composite rod.

Order	Perturbed displacements	Governing equations of influence functions
1	$u^1(x,y) = -\chi_1(y)\frac{du^0(x)}{dx}$	$\frac{d}{dy}(E^{\varepsilon}(y)\frac{d\chi_1}{dy}) = \frac{dE^{\varepsilon}(y)}{dy}$
2	$u^2(x,y) = -\chi_2(y)\frac{d^2u^0(x)}{dx^2}$	$\frac{d}{dy}(E^{\varepsilon}(y)\frac{d\chi_2}{dy}) = -2E^{\varepsilon}(y)\frac{d\chi_1}{dy} - \frac{dE^{\varepsilon}(y)}{dy}\chi_1 + E^{\varepsilon}(y) - E^H$
3	$u^3(x,y) = -\chi_3(y)\frac{d^3u^0(x)}{dx^3}$	$\frac{d}{dy}(E^{\varepsilon}(y)\frac{d\chi_3}{dy}) = -2E^{\varepsilon}(y)\frac{d\chi_2}{dy} - \frac{dE^{\varepsilon}(y)}{dy}\chi_2 - E^{\varepsilon}(y)\chi_1$

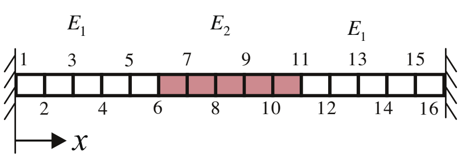


Fig. 1. A unit cell model with 15 sub elements and two clamp ends.

$$\frac{\partial}{\partial x_j}\left(E_{ijmn}^{\varepsilon}(\boldsymbol{x})\frac{1}{2}\left(\frac{\partial u_m^{\varepsilon}}{\partial x_n}+\frac{\partial u_n^{\varepsilon}}{\partial x_m}\right)\right)+f_i(\boldsymbol{x})=0 \quad \text{in} \quad \Omega\subset R^3$$
$$\boldsymbol{u}^{\varepsilon}(\boldsymbol{x})=0 \quad \text{on} \quad \partial\Omega \tag{1}$$

where E_{ijmn}^{ε} is the fourth order elastic tensor, the indices $i, j, m, n = 1, 2$ and small parameter ε indicates the proportion between the dimensions of a unit cell and the entire domain. The actual displacement u_m^{ε} in asymptotic expansion form is a function of macro and micro scales as

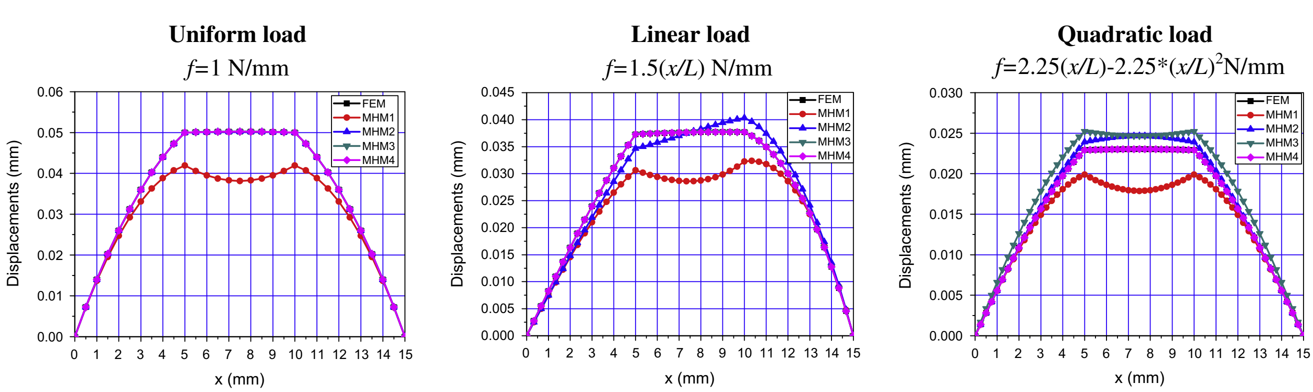
$$u_m^{\varepsilon}(\boldsymbol{x}) = u_m^0(\boldsymbol{x}) + \varepsilon u_m^1(\boldsymbol{x},\boldsymbol{y}) + \varepsilon^2 u_m^2(\boldsymbol{x},\boldsymbol{y}) + \cdots \tag{2}$$

where the homogenized displacement u_m^0 is offered by the homogenized model, and the perturbed function $u_m^j(\boldsymbol{x},\boldsymbol{y})$ of both scales is periodic in $\boldsymbol{y}$.

The perturbed displacements in separation of variables form or uncoupled form and governing differential equations for different order influence functions are listed in Table 1 wherein E^H represents the homogenized modulus. The influence functions periodic in $\boldsymbol{y}$ depends only on the scale $\boldsymbol{y}$ while the homogenized displacements and their derivatives of different orders depends only on the scale $\boldsymbol{x}$. In Table 1, the operators $\nabla_{y_j}\cdot$ and ∇_{y_j} in governing equations denotes the operation of divergence and gradient in the coordinate frame $\boldsymbol{y}$, respectively.

From the governing equations of influence function and expressions of perturbed displacements in uncoupled form, one can have following observations:

(1) The perturbed displacements in uncoupled form are defined as the multiplications of influence functions $\boldsymbol{\chi}$ and the derivatives of homogenized displacements $\boldsymbol{u}^0$. And the influence functions dependent on the micro scale $\boldsymbol{y}$ are the solutions of a unit cell problem, while the homogenized displacements dependent on the macro scale $\boldsymbol{x}$ are the solutions of global problems subjected to the given external load.
(2) The right term for solving the first order influence functions is pseudo line distributed load $\nabla_{y_j}\cdot E_{ijmn}^{\varepsilon}$ which is formed only by the material constants, the load is non-zero along boundary and interface lines of matrix and inclusion. The right term for the second order influence functions is pseudo surface load, a component of which is $E_{ipkl}^{\varepsilon} - E_{ipkl}^{H}$ which is also formed only by the material constants; but right terms for the third or higher order influence functions have not the components formed only by material constants, so we conclude that the first and second order expansion terms are necessary, and the neglect of the second order perturbation may cause unacceptable error.
(3) One can use different order of elements to calculate the influence functions and homogenized displacements since they are uncoupled or independent each other. Generally, their accuracy is different by using the same elements. One can determine the exact element order for solving influence functions from the governing equation in Table 1 when taking a periodical composite rod into account, see below.
(4) It is well known that all equations in the uncoupled MsAEM can be solved by means of the finite element method, and a salient feature of the uncoupled MsAEM is that the fine scale solution is completely described on the coarse scale. That means the actual solutions u_m^{ε} depends on the macro displacements and their derivatives $\partial\boldsymbol{u}^0/\partial\boldsymbol{x}$, $\partial^2\boldsymbol{u}^0/\partial\boldsymbol{x}^2$ and $\partial^3\boldsymbol{u}^0/\partial\boldsymbol{x}^3$, etc. if the influence functions are solved over a representative volume cell. The more micro properties can be obtained if using higher order of macro displacement derivatives.



(a) quadratic elements (b) cubic elements (c) quartic elements

Fig. 2. Displacements using different orders of elements and expansions.