Composite Structures 112 (2014) 157–168

Contents lists available at ScienceDirect

Composite Structures

journal homepage: www.elsevier.com/locate/compstruct

On explicit analytic solutions for the accurate evaluation of the shear stress in sandwich beams with a clamped end

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ARTICLE INFO

Article history:
Available online 8 February 2014

Keywords:
Sandwich structures
Warping
Saint–Venant's principle
Shear stress
Jourawski's shear theory
Finite element method

ABSTRACT

We focus on analytic solutions for the accurate computation of the shear stress in sandwich beams under flexure. To this purpose, we follow the strategy recently proposed by our group and apply the Jourawski method to the structural beam model based on the zigzag warping. We consider sandwiches whose cross-section is symmetric with respect to the neutral axis and whose skin shear deformation and core longitudinal stress are accounted for (that is, we do not limit our attention to thin skins nor to antiplane sandwiches). We obtain explicit analytic expressions for the cases of cantilever and propped-cantilever beams subject to uniform transverse load. The comparison with numerical solutions obtained through plane stress finite element simulations shows the excellent accuracy of the analytic solution, apart in a region close to the fully clamped cross-section, where the finite element solution itself is unreliable, while the new analytic solutions provide useful estimates.

1. Introduction

We focus on analytic solutions for the accurate evaluation of the shear stress in sandwich beams under bending and shear, in the linear elastic range. We restrict our attention to sandwiches whose cross-section is symmetric with respect to the neutral axis, say z, x being the beam axis. In other words, we consider sandwiches whose top and bottom layers, called the skins, are identical; however, we allow them to have *arbitrary* thickness t, that is, the skins may be *thick*, as technically defined in the classical book of Allen [1]. Also, we are interested in sandwiches in which it is allowed to disregard the core deformation ε_y along the thickness and the *relevant* Young's modulus of the core, E_c, may be large enough to influence the flexure, that is, we also account for *non-antiplane* sandwiches (see, e.g., [1,2]). Here and henceforth, the relevant Young's modulus is, for each layer of the sandwich beam, the longitudinal elastic modulus along the beam axis x, independently of the possible anisotropy of the layer.

Here, we model the sandwich beam flexure by using the structural beam theory based on the zigzag warping, formerly proposed by Yu [3] and, then, further developed by Krajcinovic [4,5] (see also [1,6,7]). Such a theory is well known to be extremely accurate in describing both the displacement field and the normal longitudinal stress (Heller [8], Sharma and Rao [9], Zuo and Hjelmstad [10], Tessler et al. [11], Tonelli et al. [2], Galuppi and Royer-Carfagni [12], Carrera et al. [13]).

The zigzag model belongs to the so-called *discrete-layer* theories [14]. However, in literature, there are other warping models based on kinematics richer than that (linear along y) the Timoshenko beam model is based on [15], often called First-Order Shear Deformation (FOSD) theory within the context of laminate structures. This second class of (Higher-Order) warping models characterises the so-called *equivalent single-layer* theories, where the displacement field $u(x, y)$ along the x direction is a suitably smooth function of y. A first reason why such theories are sometimes preferred is that they are computationally less expensive (at least for sandwiches and for laminated beams with a limited number of layers). Examples of them can be found in the works of Silverman [16], Khdeir and Reddy [17], Yu and Hodges [18], Vidal and Polit [19,20], and in the review of Ghugal and Shimpi [14].

A shortcoming affecting the sandwich beam theory based on the zigzag warping (and, in general, the *discrete-layer* theories) is that the shear stress evaluation consistent with its kinematics, computed through the constitutive equations, is, in each cross-section, uniform over each homogeneous layer, that violates the boundary and interface conditions and is far away from the actual shear stress distribution. This is a second reason why in literature *equivalent single-layer* theories have been developed. However, in laminate structures having large differences in the material properties of the layers, that is the case of sandwich structures, it is difficult for the equivalent single-layer theories to well accommodate

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http://dx.doi.org/10.1016/j.compstruct.2014.01.044
0263-8223/

the flexure behaviour. Also, Bardella and Tonelli [7] have shown how to overcome the major drawback of the zigzag theory for sandwich beams to obtain very accurate shear stress predictions: it is sufficient to apply the Jourawski [21] approximate treatment of the shear problem (originally developed to evaluate the shear stress in the classical Euler–Bernoulli beam model) to the results of the structural theory based on the zigzag warping. The analytic solutions so obtained, henceforth referred to as "Jourawski-zigzag" formulæ, are easier to deal with than exact solutions of linear elasticity (see, e.g., Pagano [22,23] for simply supported composite laminate structures). Earlier, Matsunaga [24], without explicitly mentioning Jourawski's contribution, has substantially applied the same technique, as that our group resorted to in [7], to compute the shear stress in laminated beams; in Matsunaga [24] (equivalent) point of view such a technique is obtained just by integrating the relevant continuum balance equation. However, Matsunaga [24] has adopted a Rayleigh–Ritz numerical scheme to solve the structural beam problem. Similarly, Oñate et al. [25] have recently employed the same technique as Matsunaga [24] to improve the shear stress estimate obtained by a FE implementation of the refined zigzag theory of Tessler et al. [11] for laminate composite beams with several layers. We remark that, since the improved shear stress so evaluated (i.e., either by applying Jourawski's method or by integrating the relevant continuum balance equation) turns out to be dependent on the second derivatives of the rotations of the layers [7], numerical methods may suffer lack of accuracy in estimating the shear stress. Our analytic calculations, of course, overcome this numerical issue.

The zigzag structural model employed here accounts, within each layer, for both the shear and longitudinal normal strains. The relevant kinematics is displayed in Fig. 1: it consists of the deflection $v(x)$ of the sandwich longitudinal axis x accompanied with a zigzag (piecewise linear) warping describable by two further functions $\phi_c(x)$ and $\phi_s(x)$, for the rotations of core and skins, respectively. Such a kinematics is known to quite accurately represent the actual warping of a symmetric sandwich cross-section,

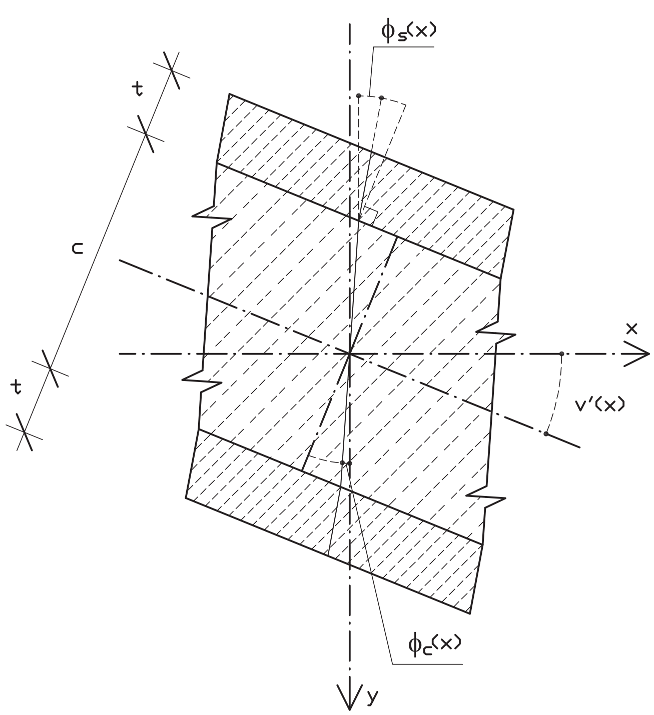

Fig. 1. Zigzag kinematics for a sandwich beam with symmetric cross-section.

and is an extension of the kinematics adopted by Allen [1], who neglected the skin shear deformation by assuming $\phi_s = v'(x)$ (here and henceforth, $'$ denotes the derivative with respect to x). Yu [3] and Krajcinovic [4,5] described the same piecewise linear kinematics for the flexural problem in terms of different terns of independent functions of x, useful to uncouple the balance differential equations obtained by minimising the Total Potential Energy (TPE) functional governing the sandwich beam flexure.

Bardella and Tonelli [7] have demonstrated the effectiveness of the Jourawski-zigzag method by solving the problem of a simply supported sandwich beam subject to uniform transverse (i.e., acting along the y direction) load q. Here, we obtain analogous analytic solutions for the cases of cantilever and propped-cantilever beams subject to uniform transverse load. These cases are particularly interesting because plane stress analyses performed with standard displacement-based Finite Element (FE) codes are generally unable to predict the stress state close to the fully clamped cross-section, even if extremely refined meshes are employed.[1] In fact, the normal longitudinal stress, σ_x, is unbounded at the corner points of the fully clamped cross-section (see, e.g., Gregory and Gladwell [26] and Tullini and Savoia [27]) and the FE solution, on the one hand, cannot describe the relevant stress field singularities and, on the other hand, violates the static boundary conditions (in terms of the transverse shear stress τ_{xy} and the normal stress σ_y) on a quite relevant part of the unconstrained sides ending in the corner points of the fully clamped cross-section. The latter problem is common to many structures modelled as continua and having a fully clamped side with corners (see, e.g., the three-dimensional analysis of a circular functionally-graded plate by Sburlati and Bardella [28]). As analytically demonstrated in Section 2.1, also the Jourawski-zigzag estimates at fully clamped cross-sections seem to suffer some limitations. Such limitations, however, do not lead to unrealistic shear stress estimates. In fact, after presenting the theoretical model in Section 2, in Sections 3 and 4, we will show that there is an excellent agreement between our analytic predictions and the results of very refined plane stress FE analyses, apart from a region very close to the fully clamped cross-section, where our analytic predictions, provided explicitly in subSections 3.1.2 and 4.1.2, satisfy all the boundary and interface conditions and are useful estimates, obtained without the need of dealing with the longitudinal stress singularity. In order to give ground to our conclusions (Section 5), we will consider a sandwich cross-section of high *heterogeneity*, that is, having very thick skins and elastic moduli of the skins much larger than those of the core. By the way, in Section 3, we will show that such a choice results in a counterintuitive deformed shape obtained for the cantilever problem, very well described by the analytical model.

For both problems, is Sections 3.1.1 and 4.1.1, we will also provide the explicit analytic expressions for the shear stresses on the neutral axis and on the interface, the latter of crucial importance since governing the delamination.

2. Structural theory based on the zigzag kinematics plus Jourawski-like shear stress evaluation

Assuming the zigzag warping represented in Fig. 1, the TPE functional Ψ for a sandwich beam of length L subject to a transverse distributed load $q(x)$ reads:

[1] Preferably, we will indicate as *fully clamped* cross-section any beam end where all the kinematics variables are set to zero. This terminology is in fact reminiscent of the analogous constraint in the corresponding continuum plane stress boundary value problem, in which both displacement components are *pointwise* set to zero.