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# Nonlocal nonlinear free vibration of functionally graded nanobeams

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ABSTRACT

In this paper, nonlinear free vibration of functionally graded (FG) nanobeams with immovable ends, i.e. simply supported-simply supported (SS) and simply supported-clamped (SC), is studied using the nonlocal elasticity within the frame work of Euler–Bernoulli beam theory with von kármán type nonlinearity. The material properties are assumed to change continuously through the thickness of the FG nanobeam according to a power-law distribution. The analytical solution for the nonlinear natural frequency is established using the method of multiple scale. The small scale effects on the linear/nonlinear nonlocal frequency to the linear/nonlinear classical frequency ratios (the linear/nonlinear frequency ratios) are examined for various parameters such as the FG nanobeam length, the FG nanobeam thickness to length ratio (the thickness ratio), the vibration amplitude to the radius of gyration ratio (the amplitude ratio), and the boundary condition. As a main result, it is observed that while the linear frequency ratios are independent of the gradient index, the nonlinear frequency ratios vary with the gradient index.



## 1. Introduction

Due to superior properties, nanostructures have attracted much attention in the recent years. Multiple recent experimental results have shown that as the size of the structures reduces to micro/nanoscale, the influences of atomic forces and small scale play a significant role in mechanical properties of these nanostructures [1–3]. Thus, neglecting these effects in some cases may results in completely incorrect solutions and hence wrong designs. The classical continuum theories do not include any internal length scale. Consequently, these theories are expected to fail when the size of the structure becomes comparable with the internal length scale. Eringen nonlocal theory is one of the well-known continuum mechanics theories [4–7] that includes small scale effects with good accuracy to model micro/nanoscale devises. The nonlocal elasticity theory assumes that the stress at a point is function of the strain at all neighbor points of the body, hence, this theory could take into account the effects of small scales.

In recent years, the studies of nanostructures using the nonlocal elasticity theory have been an area of active research. Based on this theory, Reddy [8] derived the equation of motion of various kinds of beam theories available (Euler–Bernoulli, Timoshenko, Reddy and Levinson) and reached analytical and numerical solutions on static deflections, buckling loads, and natural frequencies. Various nonlocal beam theories are used for bending, buckling, post-buckling, linear transverse and longitudinal vibration, and instability analyses of single- and multi- walled carbon nanotubes (CNTs) [9–23]. Eltaher et al. [24] presented free vibration analysis of functionally graded size-dependent nanobeams using finite element method based on the Euler–Bernoulli beam theory. As applications of the nonlocal elasticity theory in nonlinear analysis, Reddy [25] derived nonlocal nonlinear formulation for bending of classical and shear deformation theories of beams and plates but he did not present any numerical results. Yang et al. [26] and Ke et al. [27] used nonlocal Timoshenko beam theory for nonlinear free vibration analysis of single- and embedded double-walled carbon nanotubes, respectively. In similar works, Ghorbanpour et al. [28] considered nonlinear vibration of embedded single-walled boron nitride nanotubes (SWBNNTs) based on nonlocal Timoshenko beam theory, and Ansari and Ramezannezhad [29] investigated the large-amplitude vibrations of embedded multi-walled carbon nanotubes including thermal effects using nonlocal Timoshenko beam model. The effect of small scales on wave propagation of single- and multi-walled carbon nanotubes was also considered [30–38].

As seen, there is no study investigating the small scale effect on nonlinear static and dynamic analyses of functionally graded nanobeams, while it is necessary to be familiar with the mechanical behavior of FG nanoscale structures for nano/micro-electro-mechnaical systems (NEMS/MEMS) fabrication. The main purpose of the present work is to propose a comprehensive analytical model to study the small scale effects on the nonlinear free vibration of nanoscale FG Euler–Bernoulli beams with von kármán type nonlinearity. To this end, the equation of motion is obtained

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and the analytical solution for nonlinear natural frequency is established using the method of multiple scale. The small scale effects on the frequency ratios of FG nanobeams are examined for various boundary conditions, nanobeam lengths, amplitude ratios, and thickness ratios.

## 2. Formulation

Consider a FG nanobeam with length $L$ ($0 \leqslant x \leqslant L$), thickness $h$ ($-0.5h \leqslant z \leqslant 0.5h$), and width $b$ ($-0.5b \leqslant y \leqslant 0.5b$). The FG nanobeam is generally composed of two different materials at the top and the bottom surfaces (as shown in Fig. 1). Poisson's ratio $v$ is assumed to be constant, i.e. $v = 0.3$, whereas bulk elastic modulus $E(z)$ and mass density $\rho(z)$ are assumed to vary in the thickness direction according to the power law distribution:

$$E(z) = (E_1 - E_2)\left(\frac{2z+h}{2h}\right)^m + E_2 \tag{1}$$

$$\rho(z) = (\rho_1 - \rho_2)\left(\frac{2z+h}{2h}\right)^m + \rho_2 \tag{2}$$

where the subscripts 1 and 2 denote the top surface and the bottom surface, respectively, and a gradient index $m$ determines the variation profile of material properties across the FG nanobeam thickness. Upon the Euler–Bernoulli beam model, the displacement field at any point of the nanobeam can be written as

$$u_x(x,z,t) = U(x,t) - z\frac{\partial W(x,t)}{\partial x} \tag{3}$$

$$u_z(x,z,t) = W(x,t) \tag{4}$$

where $U(x,t)$ and $W(x,t)$ are the displacement components of the mid-plane at time $t$. In accordance, the von kármán type nonlinear strain–displacement relationship is

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} + \frac{1}{2}\left(\frac{\partial u_z}{\partial x}\right)^2 = \frac{\partial U}{\partial x} - z\frac{\partial^2 W}{\partial x^2} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2 \tag{5}$$

Now, using Hamilton's principle, the nonlinear equations of motion of the nanobeam can be derived as

$$\frac{\partial N_{xx}}{\partial x} = I_1\frac{\partial^2 U}{\partial t^2} \tag{6}$$

$$\frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial W}{\partial x}\right) = I_1\frac{\partial^2 W}{\partial t^2} \tag{7}$$

where $N_{xx}$ and $M_{xx}$ are the local force and bending moment resultants, respectively, and given by

$$N_{xx} = \int_A \sigma_{xx}dA = bA_1\left(\frac{\partial U}{\partial x} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2\right) - bB_1\left(\frac{\partial^2 W}{\partial x^2}\right) \tag{8}$$

$$M_{xx} = \int_A z\sigma_{xx}dA = bB_1\left(\frac{\partial U}{\partial x} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2\right) - bC_1\left(\frac{\partial^2 W}{\partial x^2}\right) \tag{9}$$

and parameters used in Eqs. (8) and (9) are defined as

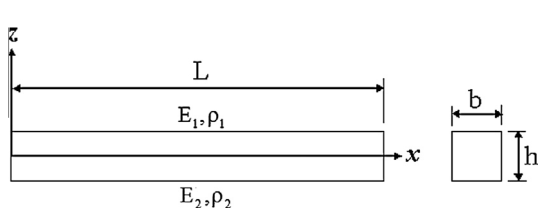


**Fig. 1.** Geometry of a FG nanobeam.

$$\{A_1, B_1, C_1\} = \int_{-h/2}^{+h/2} E(z)\{1, z, z^2\}dz, I_1 = \int_{-h/2}^{+h/2} b\,\rho(z)\,dz \tag{10}$$

Equations of motions, Eqs. (6) and (7), can be used in nonlocal form as follows

$$\frac{\partial N_{xx}^{nl}}{\partial x} = I_1\frac{\partial^2 U}{\partial t^2} \tag{11}$$

$$\frac{\partial^2 M_{xx}^{nl}}{\partial x^2} + \frac{\partial}{\partial x}\left(N_{xx}^{nl}\frac{\partial W}{\partial x}\right) = I_1\frac{\partial^2 W}{\partial t^2} \tag{12}$$

If the axial inertia is neglected, Eq. (11) gives

$$N_{xx}^{nl} = N_0 = \text{Constant} \tag{13}$$

The nonlocal force and bending moment resultants can be obtained by multiplying the left-hand side of Eqs. (8) and (9) by $(1 - \mu\nabla^2)$ [7], using Eqs. (11)–(13) and doing some mathematical manipulation; and they are given by

$$N_{xx}^{nl} = bA_1\left(\frac{\partial U}{\partial x} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2\right) - bB_1\left(\frac{\partial^2 W}{\partial x^2}\right) \tag{14}$$

$$M_{xx}^{nl} = \mu\left(-N_{xx}^{nl}\frac{\partial^2 W}{\partial x^2} + I_1\frac{\partial^2 W}{\partial t^2}\right) + bB_1\left(\frac{\partial U}{\partial x} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2\right) - bC_1\left(\frac{\partial^2 W}{\partial x^2}\right) \tag{15}$$

In order to express the bending moment in terms of deflection, Eq. (14) can be rewritten as follow

$$bB_1\left(\frac{\partial U}{\partial x} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2\right) = \frac{B_1}{A_1}\left(N_{xx}^{nl} + bB_1\left(\frac{\partial^2 W}{\partial x^2}\right)\right) \tag{16}$$

Substituting Eq. (16) into Eq. (15) leads to

$$M_{xx}^{nl} = \mu\left(-N_{xx}^{nl}\frac{\partial^2 W}{\partial x^2} + I_1\frac{\partial^2 W}{\partial t^2}\right) + \frac{B_1}{A_1}\left(N_{xx}^{nl} + bB_1\left(\frac{\partial^2 W}{\partial x^2}\right)\right) - bC_1\left(\frac{\partial^2 W}{\partial x^2}\right) \tag{17}$$

For nanobeams with immovable ends (i.e. $U$ and $W = 0$, at $x = 0$ and $L$) and with Eq. (13) in mind, integrating Eq. (14) with respect to $x$ leads to

$$N_{xx}^{nl} = N_0 = \frac{bA_1}{2L}\int_0^L\left(\frac{\partial W}{\partial x}\right)^2dx - \frac{bB_1}{L}\int_0^L\left(\frac{\partial^2 W}{\partial x^2}\right)dx \tag{18}$$

Finally, substitution of Eqs. (17) and (18) into Eq. (12) gives the nonlocal nonlinear governing equation for the Euler–Bernoulli functionally graded nanobeam as follows

$$(P)\frac{\partial^4 W}{\partial x^4} + (K)\frac{\partial^2 W}{\partial x^2} - I_1\frac{\partial^2 W}{\partial t^2} + \mu I_1\frac{\partial^4 W}{\partial x^2\partial t^2} = 0 \tag{19}$$

where

$$P = \frac{bB_1^2}{A_1} - bC_1 - \mu\frac{bA_1}{2L}\int_0^L\left(\frac{\partial W}{\partial x}\right)^2dx + \mu\frac{bB_1}{L}\int_0^L\left(\frac{\partial^2 W}{\partial x^2}\right)dx$$

$$K = \frac{bA_1}{2L}\int_0^L\left(\frac{\partial W}{\partial x}\right)^2dx - \frac{bB_1}{L}\int_0^L\left(\frac{\partial^2 W}{\partial x^2}\right)dx$$

The nonlinear equation of motion of the conventional Euler–Bernoulli FG beam [39] can be obtained from Eq. (19) by setting $\mu = 0$.