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Axiomatic/asymptotic PVD/RMVT-based shell theories for free vibrations of anisotropic shells using an advanced Ritz formulation and accurate curvature descriptions

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The Hierarchical Trigonometric Ritz Formulation (HTRF) has earlier been successfully developed for plates [1–4] and shells [5] using the Principle of Virtual Displacements (PVD). In this paper the HTRF is significantly extended with the help of Reissner's Mixed Variational Theorem (RMVT) so as to deal with the free vibrations of doubly-curved anisotropic laminated composite shells. The interlaminar equilibrium of the transverse normal and shear stresses is fulfilled a priori by exploiting the use of Lagrange multipliers. The transverse normal and shear stresses thus become primary variables within the formulation and are always modeled with a Layer-Wise kinematics description. Equivalent Single Layer, Zig-Zag and Layer-Wise approaches are instead efficiently used for the displacement primary variables. Appropriate expansion orders for each displacement or stress unknown are selected depending on the required accuracy and the computational cost. Axiomatic/asymptotic shell theories are then developed by virtue of a deep study on the effectiveness of each term both in the displacements and in the transverse stresses fields. Next exact and/or accurately approximated curvature descriptions are taken into account. Cylindrical, spherical and hyperbolic paraboloidal shells are investigated. The proposed advanced quasi-3D shell models are assessed by comparison with 3D elasticity solutions.

1. Introduction

A significant part of aerospace structures is generally made up of thin-walled cylindrical or spherical shell components. Such application is justified because of the extraordinary load-carrying capability of shell structures. Furthermore, their structural efficiency is characterized by a high stiffness-to-weight and strength-to-weight ratios. Clearly cylindrical and spherical panels widely used in the latest aerospace applications require a deep and comprehensive investigation for their dynamic behavior. The majority of the shell theories which have been used in the last decades are inadequate to successfully address this task and the use of advanced shell models is needed. Tracing-back the history of the shell theories, according to Novozhivol [6] the first known effort properly channeled to shell theory was given by Aron [7], who essentially tried, for the first time, to extend Kirchhoff's hypotheses, valid for flat plates, to shell structures. Despite the pioneering attempt his development was not strictly correct and some inaccuracies were rectified by Love [8,9]. Nevertheless even the development of the theory proposed by Love [8,9] was featured by some mathematical inconsistencies dwelling in the fact that some small terms were retained whereas some others of the same order of magnitude were discarded. The first set of Governing Differential Equations (GDEs) completely elucidated and free from inconsistencies were provided by Lur'e [10]. Gol'denveizer [11], for the first time, provided the compatibility conditions of strains for shells starting from the Gauss-Codazzi conditions. In his work, Gol'denveizer introduced pioneering insight into the possibility of identically satisfying the GDEs, written in terms of forces and moments, by virtue of stress functions. Contemporaneously and in a completely independent manner from Gol'denveizer [11], Lur'e [10] provided a similar solution. More specifically both authors, considering a shell loaded only at the edge, proved that the GDEs made up of ten unknowns including forces and moments could have been expressed in terms of only four arbitrary stress functions. This solution was somehow an extension to shell structures of the solution provided by Airy [12] in the plane theory of elasticity. Afterwards, Novozhilov [6], by setting the Poison's ratio to zero, provided several complex forms of the GDEs written in a compact and concise manner, showing that the advantages of this new formulation lie in the simplification of their solution. A deeper understanding of the usefulness of the application of the complex transformations was given by Mushtari [13,14]. Subsequently, in order to provide solutions of the GDEs of practical interest, considerable efforts were focused and devoted to their simplification. As a result, the GDEs of shallow shells were

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derived. In this regard, it worths highlighting the articles of Donnell [15,16], Mushtari [13,14] and Vlasov [17,18]. Most notably, the turning point was given by Donnell [15], who wrote: *"Much confusion seems to exist as to what simplification can be made, and the condition under which they can be made. One author consider items which another reject and vice versa. An attempt is made, in the following discussion, to clarify this question and to obtain the greatest simplification possible, under the condition of the present problem; the results are applicable to a large class of problems"*. His efforts led to the theory universally adopted for the investigation of shallow shell structures. Afterwards, independently each other, Flügge [19,20], Lur'e [10] and Byrne [21] developed a theory discarding the hypothesis of thinness considering higher order expansion of the reciprocal of the Lamé parameters. Other refinements of the developed shell theories were proposed by Sanders [22]. Additional effects in the development of shell theories were taken into account by Whitney and Sun [23], Librescu [24], Gulati and Essemberg [25], Zucas and Vinson [26] and Ambartsumian [27–34] amongst others. Additional references can be found in Naghdi [35], Ambartsumian [36] and Bert [37–39]. Reddy [40] proposed a generalization of Sander's theory to anisotropic doubly-curved shells. The application of Layer-Wise theories for shell structures can be found in the papers presented by Hsu and Wang [41], Cheung and Wu [42], Barbero et al. [43] and Carrera [44–46]. Reviews on finite element shell formulations have been given by Denis and Palazzotto [47] and Di and Ramm [48]. Exhaustive reviews on classical theories can be found in Librescu [24]. As regards the use of approximation methods, Qatu and Asadi [49] addressed the vibration analysis of doubly-curved shallow shells with arbitrary boundary conditions by using the Ritz method with algebraic polynomial displacement functions. Asadi et al. [50] employed a 3D and several shear deformation theories in order to carry out static and vibration analysis of thick deep laminated cylindrical shells. Ferreira et al. [51] used a wavelet collocation method for the analysis of laminated shells. The same author [52] combined a sinusoidal shear deformation theory with the radial basis functions collocation method to deal with static and vibration analyzes of laminated composite shells. Tornabene et al. [53,54] studied the free vibration behavior of doubly-curved anisotropic laminated composite shells and revolution panels by means of the Generalized Differential Quadrature (GDQ). Nevertheless the majority of the papers touched upon hitherto do not describe completely what have been referred to as C_z^0-requirements by Carrera [55]. Most notably, due to the high transverse shear deformability of composite structures usually in their advanced modeling continuity of both displacements and transverse stresses is required. With shell models based on the use of the Principle of Virtual Displacements (PVD) the C_z^0-requirements are not fulfilled because it does not account for the interlaminar continuity (IC) of the transverse stresses. To completely overcome this drawback, Reissner's Mixed Variational Theorem (RMVT) [56–60] is invoked in this paper in the derivation of both weak and strong forms of the governing equations. By means of RMVT the IC is fulfilled a priori by exploiting the use of the Langrange multipliers which allow to variationally enforce the compatibility of the transverse shear and normal strains. Moreover due to the strong transverse anisotropy showed by composite structures, advanced shell models must include the Zig-Zag (ZZ) trend of the displacement components through the thickness direction. Theories regarding the fulfillment of the C_z^0-requirements were reviewed by Grigolyuk and Kulikov [61]. Within the framework of static and dynamic analysis of composite shell structures the application of asymptotic methods must not be underrated and in particular the works of Fettahlioglu and Steel [62], Widera and Logan [63], Widera and Fan [64], Spencer et al. [65] and Cicala [66] worth to be highlighted. A complete overview of different problems related to multilayered shells modeling has been provided by Kapania [67] and Noor and Burton [68]. The first purpose of the present paper is to evaluate the effectiveness of the higher order terms for both displacements and transverse stresses fields. This procedure leads to the so-called axiomatic/asymptotic theories. The mathematical formulation which permits the application of this new *modern approach* in the dynamic analysis of advanced structures, essentially based on the combination of Lagrangian mechanics and advanced variational principles, is the Carrera Unified Formulation (CUF) [69–71] which is extensively used in the present paper. In particular, the first author has developed a Matlab code which includes the PVD/RMVT-based HTRF with the capability to generate axiomatic/asymptotic theories within the framework of the CUF. It is worth underscoring that the PVD/RMVT-based HTRF is generated by the combination of the CUF, the trigonometric Ritz method and the PVD or RMVT variational statements. The second purpose is to clarify some existing doubts and controversial questions on the dichotomy between the use of exact (or accurately approximated) curvature terms or higher order Equivalent Single Layer (ESL), ZZ and Layer-Wise (LW) shell theories in the analysis of doubly-curved shell structures. Results are presented in terms of natural frequencies, the effects of stacking sequence, length-to-thickness ratio, radius-to-length and radius-to-thickness ratios are analyzed. Several curvature approximations are tested towards the exact one. Finally some conclusions are drawn from the findings of the research.

2. Laminated composite shell geometries

The shell geometries investigated in this paper are shown in Fig. 1 and the parameters used to describe each shell geometry are depicted in Fig. 2. The generic laminated shell is composed of N_l layers. Subscripts and superscripts k refer to the layer number which starts from the bottom of the shell. The layer geometry is denoted by the same symbols as those used for the whole multilayered shell and vice versa. With α_k and β_k the curvilinear orthogonal coordinates (coinciding with the lines of principal curvature) on the layer reference surface Ω_k (middle surface of the k layer) are indicated. The z_k denotes the rectilinear coordinate in the normal direction with respect to the layer middle surface Ω_k. The Γ_k is the Ω_k boundary: Γ_k^g and Γ_k^m are those parts of Γ_k on which the geometrical and mechanical boundary conditions are imposed, respectively. These boundaries are herein considered parallel to α_k or β_k. For convenience the dimensionless thickness coordinate $\zeta_k = \frac{2\,z_k}{h_k}$ (h_k denotes the thickness in A_k domain) is introduced. Most notably, in Fig. 2, $\boldsymbol{r}(\alpha, \beta)$ indicates the position vector of a point on the middle surface Ω of the shell, $\boldsymbol{R}(\alpha, \beta, z)$ is the position vector of a generic point within the volume occupied by the shell. At each point P of the middle surface $\boldsymbol{n}(\alpha, \beta)$ indicates the unit normal vector. The following relationships hold in the given orthogonal system of curvilinear coordinates [44,45]:

1. Square of line elements

$$ds_k^2 = \left(H_\alpha^k\right)^2 \, d\alpha_k^2 + \left(H_\beta^k\right)^2 \, d\beta_k^2 + \left(H_z^k\right)^2 \, dz_k^2 \tag{1}$$

2. Area of an infinitesimal rectangle on Ω_k

$$d\Omega_k = H_\alpha^k \, H_\beta^k \, d\alpha_k \, d\beta_k \tag{2}$$

3. Infinitesimal volume

$$dV_k = H_\alpha^k \, H_\beta^k \, H_z^k \, d\alpha_k \, d\beta_k \, dz_k \tag{3}$$

where

$$H_\alpha^k = A^k \left(1 + \frac{z_k}{R_\alpha^k}\right) \qquad H_\beta^k = B^k \left(1 + \frac{z_k}{R_\beta^k}\right) \qquad H_z^k = 1 \tag{4}$$