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Hybrid equilibrium finite element formulation for composite beams with partial interaction

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ABSTRACT

Thanks to their various benefits, composite beams have been increasingly used in various applications. This study will focus on two-layer composite beams with a flexible shear interface between layers. The finite element method, in particular its displacement-based formulation, has been recognized as the most popular method for numerical analysis of composite beams. However, when applied to Timoshenko beams with partial interaction, the displacement-based formulation may suffer from the so-called *shear-locking* and *slip-locking* phenomena, leading to erroneous solutions. Hybrid and mixed finite element formulations have been viewed as competitive alternatives, since they naturally avoid locking effects. Special types of these formulations are the so-called *equilibrium-based formulations*, producing statically admissible solutions. This work introduces for the first time an equilibrium-based finite element formulation for the analysis of Timoshenko composite beams with partial interaction. The formulation relies on a variational principle of complementary energy involving only force/moment-like variables as fundamental unknown fields. The approximate field variables are selected such that all equilibrium equations hold in strong form. The inter-element equilibrium is enforced by resorting to the Lagrangian multiplier method. Unlike traditional displacement-based finite element formulations, the proposed scheme is naturally free from both shear- and slip-locking phenomena. The accuracy and effectiveness of the new formulation is numerically assessed through the analysis of several numerical examples. In particular, the ability of the formulation to model accurately both very flexible and very stiff shear connections is numerically shown.

1. Introduction

Composite materials have a number of advantages over their isotropic counterparts, including low density, high strength and stiffness, as well as tailoring their properties by changing fibre orientation. Composite beams have been increasingly used in various industries, such as aerospace, automotive, nuclear, marine, biomedical, and civil engineering. Composite sandwich and laminated beams are typical components used in aerospace structures, while in civil engineering, steel–concrete composite beams in buildings and bridges, wood-concrete floor systems, and concrete beams externally reinforced with laminates are often encountered. This study focuses on two-layer composite beams with a single flexible shear interface, such as steel–concrete beams or two-layer wooden beams.

The mechanical behaviour of composite structures depends to a large extent on the character of bonding. If the layers are connected continuously by means of strong adhesives, the mechanical assumption of a perfect bond between the layers is reasonable. However, the layers are often connected non-continuously, by means of connectors, such as shear studs and nails, which are not rigid. Therefore, some slip and uplift can occur at the interlayer. While the uplift is often small and can be neglected, the interlayer slip significantly affects the behaviour of composite elements. This phenomenon is called *partial* (or *incomplete*) *interaction* and is an important issue in composite structures [24]. In fact, the inclusion of the interlayer-slip effect in the theory of composite beams is essential for optimal design and accurate representation in simulations of the actual mechanical behaviour of composite structures with partial interaction. Many efforts and a large number of research studies have been devoted to obtain the solution to this problem.

The first one-dimensional composite beam model with flexible shear connectors was developed by Newmark et al. [23], in which two layers were assumed to be connected in such a way that vertical separation did not occur between the components. Both layers were assumed to follow the kinematic assumptions of the Euler–Bernoulli beam theory. Fairly recently, analytical solution methods based on Newmark's model for the static response of two-layer beams with interlayer slip in the linear-elastic regime were proposed, e.g., in [39,15,19,28,27,16]. All these studies are based on the Euler–Bernoulli beam theory, which considers that planes that are perpendicular to the beam axis before bending will remain

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plane and perpendicular to it after deformation. In other words, the Euler–Bernoulli beam theory does not consider any effect of transverse shear deformation. However, for beams either with a small span-to-depth ratio, low shear rigidity, or continuous spans, the effect of transverse shear deformation is not small and, therefore, cannot be disregarded. The shear deformation effect was incorporated in the analysis for the first time by Murakami [21], who used Timoshenko's (also often referred to as first-order) beam theory to represent the deformation of the beam layers. Analytical solutions for two-layer beam elements based on the first-order shear deformation theory, in which different shear deformations were allowed in the two layers, were derived in [36]. A formulation for the analysis of members with non-uniformly distributed shear connectors and interlayer transverse separation was proposed in [6]. For a review on various Euler–Bernoulli and Timoshenko-based models that were proposed in the literature for the analysis of composite beams with partial interaction the reader is referred to [20].

Composite elements based on (more sophisticated) higher-order shear deformation beam theories, which incorporate the warping of the beam section produced by shear deformation by taking a non-linear variation of the axial displacement of the fibres over the beam depth, were proposed in, e.g., [43,11,1,38,22,5]. Numerical methods, in particular the finite element method, have also been widely used in the analysis of both linear and non-linear composite beams with partial interaction. Displacement-based Timoshenko finite element formulations for the analysis of partial interaction composite members were developed in, e.g., [2,10,7]. However, these formulations may suffer from the shear-locking and slip-locking phenomena.

As is well known, classical displacement-based Timoshenko beam elements with the same interpolation of both the transverse and rotation fields behave very stiff in the thin-beam limit, i.e., as the length-to-thickness ratio becomes large. Such a behaviour is known as shear-locking and is due to the inconsistency of the interpolation for the transverse and rotation fields. To overcome shear-locking, one may use equal interpolation for both fields, but use a lower-order polynomial for the shear strain. This is often realized by using selective integration, in which reduced-order integration is used to evaluate the stiffness coefficients associated with the transverse shear strain, and all other coefficients of the stiffness matrix are evaluated using full integration. Unfortunately, this procedure leads to spurious energy modes [26]. Alternative approaches based on hybrid and mixed finite element formulations to alleviate shear-locking were pursued by many investigators, e.g., [18,25,37,17]. While the latter are based on multi-field variational principles, such as a two-field Hellinger–Reissner principle or a three-field Hu-Washizu principle, the former rely on modified variational principles with relaxed continuity requirements across element boundaries, thus assuming independent approximations for field variables within the element and along the boundaries.

As for the slip-locking phenomenon, it occurs due to the coupling between the transverse and axial displacement fields and it can lead to erroneous oscillations in the slip field and a considerable reduction of the optimal rate of convergence for high values of the connection stiffness [7,8]. Models that attempt to overcome these limitations within the framework of composite members with partial interaction were proposed in [30,4,9,3,36]. Alternative strategies to alleviate the slip-locking behaviour in the classical displacement-based finite element formulation were recently adopted in [14], where techniques based on the assumed strain method, discrete strain gap method, and kinematic interpolatory method were introduced.

Hybrid and mixed finite element formulations can be used to naturally avoid locking effects, without a need to resort to any numerical tricks. Special types of these formulations are the so-called equilibrium-based formulations, first introduced in [12,13] for small elastic deformation problems. They are very often derived from complementary variational principles. In these formulations, the approximate fields are chosen so that the stress fields are in equilibrium or, in other words, internal equilibrium and continuous stress transmission between elements are satisfied exactly. In contrast, the compatibility differential equations and the Dirichlet boundary conditions are only satisfied in a weak form. The numerical solutions obtained with these models are called *statically admissible solutions*. These formulations have a special appeal for practical design engineers, despite the popularity of the conventional displacement formulations, due to the exact transmission of stresses across boundaries between adjacent structural members. This avoids the need for the 'averaging' procedures required to obtain unique nodal values of stresses when resorting to displacement formulations. In fact, in structural engineering design, the stresses are often the variables of most interest, whereas the displacements are of secondary interest. Indeed, although displacement formulations can lead to sufficiently accurate displacement fields, the corresponding stress fields may be highly erroneous. This occurs since the accuracy of the approximate displacement field rapidly deteriorates when differentiations are required to compute other results, such as stresses or strains. In contrast, for equilibrium formulations, the stresses are computed as fundamental unknowns. Examples of equilibrium-based finite element formulations for geometrically non-linear beam problems were presented in [33,35,32]. An equilibrium-based formulation for non-linear elastic cables can be found in [34]. For further details on these formulations the reader is referred to [31].

A finite element formulation for non-linear composite beams with partial interaction considering only forces and moments as fundamental unknowns that satisfy *a priori* the equilibrium differential equations was proposed in [30,29]. However, besides being based on the Euler–Bernoulli beam theory, disregarding therefore shear deformation effects, such a formulation leads to slip distributions that are not continuous across the inter-element boundaries. A mixed formulation also for non-linear Euler–Bernoulli-based composite beams was introduced in [4]. Such a formulation not only uses two types of approximations, i.e., displacements and internal forces/moments, which require to comply with additional stability conditions, it also leads to solutions that do not satisfy the equilibrium differential equations of the problem in a strong form.

In this work, a hybrid equilibrium-based finite element formulation for the analysis of composite Timoshenko beams with partial interaction is introduced for the first time. This formulation relies on a variational principle of complementary energy only involving force- and moment-like variables as fundamental unknown fields, and leads to statically admissible solutions, i.e., solutions that satisfy all the equilibrium conditions in a strong form. Unlike traditional displacement-based finite element formulations, the proposed scheme is naturally free from both shear- and slip-locking phenomena. Feasibility and effectiveness of the proposed formulation is numerically demonstrated through the analysis of several numerical tests.

2. Boundary-value problem

The aim of this study is to investigate the behaviour of composite beams with two material layers with a shear flexible interface as shown in Fig. 1. As discussed in the previous section, the Timoshenko (or first-order shear deformation) theory is adopted to describe the deformation of the beam layers. Hence, transverse shear deformations are allowed, and the rotation angle and shear deformations are assumed to be identical in the two layers. In addition, the following assumptions are considered: (i) no uplift occurs between the two layers (i.e., both layers have the same transverse displacement); (ii) slip can occur at the interlayer (i.e., partial interaction is assumed); (iii) the layers are connected continuously