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Approximate analytical solutions for the nonlinear free vibrations of composite beams in buckling

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ABSTRACT

We present approximate analytical solutions for the nonlinear free vibrations of symmetrically or asymmetrically laminated composite beams in prebuckling and postbuckling. Simply supported and clamped–clamped boundary conditions are considered. Galerkin's discretization is used to obtain the nonlinear ordinary differential equations governing the large-amplitude vibrations of composite beams in prebuckling and postbuckling, which are found to be of the same form. The variational method of He [20,21] is used to derive an approximate analytical solution for the nonlinear natural frequency and the nonlinear load–deflection relation. Results obtained by using the proposed analytical solution is compared with the finite element results available in the literature and a good agreement has been obtained. Numerical results to show the variation of the nonlinear natural frequency with the applied axial load for a variety of composite laminates are presented. The contribution of the amplitude of vibration on the nonlinear load–deflection response and the nonlinear natural frequency is found to be significant.

1. Introduction

Fiber-reinforced laminated composite structures are used in many engineering applications due to their superior properties such as high specific strength and high specific stiffness. Structures that carry inplane loadings are subject to buckle. The critical buckling load defines the threshold at which the prebuckling equilibrium position loses its stability. For beams, it is shown that the first buckling mode is the only stable equilibrium position, in the postbuckling state, and all higher order modes are unstable, Nayfeh and Emam [1]. This means that beams have a load-carrying capacity in postbuckling as well. Linear free vibration analysis is a basic element of understanding the dynamic response of a structure that is subjected to dynamic loadings. On the other hand, under severe dynamic loading conditions, structures may undergo large-amplitude vibrations. In this case, linear free vibration analysis will not be adequate and a nonlinear free vibration analysis becomes necessary. The nonlinear free vibrations of isotropic beams have received a considerable attention by many researchers [2–9]. On the other hand, a few studies have been reported on the nonlinear free vibrations of composite beams [10–14]. Gunda et al. [15] studied large-amplitude vibrations of laminated composite beam with axially immovable ends with symmetric and asymmetric layup orientations by using the Rayleigh–Ritz (R–R) and finite element methods. Geometric nonlinearity of von-Karman type, which accounts for the midplane stretching, is considered. Results presented in that study are valid as long as the beam in its prebuckling state. Baghani et al. [16] presented analytical expressions for large-amplitude free vibration and post-buckling analysis of unsymmetrically laminated composite beams on elastic foundation. Besides, the elastic foundation has cubic nonlinearity with shearing layer. The nonlinear governing equation is solved by employing the variational iteration method. They presented the effects of different parameters on the ratio of nonlinear to linear natural frequency and the post-buckling load–deflection relation. Their analysis was also valid in the prebuckling state.

To the best of author's knowledge, the nonlinear free vibrations of composite beams in the postbuckling state has not been addressed yet, which was the motivation behind this study. The main objective of this study is to present an approximate analytical solution for the nonlinear free vibrations of laminated composite beams in postbuckling using He's variational principle. Simply supported and clamped–clamped boundary conditions are used. The equation governing the large-amplitude vibrations of composite beams is a nonlinear integral partial-differential equation. A single-mode Galerkin discretization is used to reduce the governing equation into a nonlinear ordinary-differential equation. It is found out that the equations governing the nonlinear free vibrations of composite beams in prebuckling and postbuckling have a similar form. The model is validated by comparing present results with

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the finite element results available in the literature where a good agreement is obtained. Numerical results show that the amplitude of vibration has a significant effect on the nonlinear natural frequency in prebuckling and postbuckling.

2. Problem definition

We consider a composite laminated beam of length ℓ, height h, width b, and mass density ρ that is subjected to a compressive axial load $\hat{P}$. The nondimensional equation governing free, undamped, large-amplitude lateral vibrations measured from the undeformed equilibrium position is given by [17]

$$\ddot{w} + w^{iv} + \left(P - \frac{1}{2}\alpha\int_0^1 w'^2 dx\right)w'' + \Lambda[w'(1,t) - w'(0,t)]w'' = 0 \tag{1}$$

where

$$x = \frac{\hat{x}}{\ell},\ w = \frac{\hat{w}}{r},\quad t = \hat{t}\sqrt{\frac{\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)}{\rho h \ell^4}},\ P = \frac{\hat{P}\ell^2}{b\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)},$$
$$\alpha = \frac{A_{11}r^2}{D_{11} - \frac{B_{11}^2}{A_{11}}}, \Lambda = \frac{B_{11}r}{D_{11} - \frac{B_{11}^2}{A_{11}}} \tag{2}$$

are nondimensional parameters. The dot indicates the derivative with respect to time t, the prime indicates the derivative with respect to the spatial coordinate x, and the hat identifies dimensional quantities. Here r is the radius of gyration of the cross section ($r = \sqrt{I/A}$), A_{11}, B_{11}, and D_{11} are, respectively, the axial, coupling, and bending stiffnesses defined as

$$A_{11} = \sum_{k=1}^{N}\overline{Q}_{11_k}(\hat{z}_k - \hat{z}_{k-1}) \tag{3}$$

$$B_{11} = \frac{1}{2}\sum_{k=1}^{N}\overline{Q}_{11_k}(\hat{z}_k^2 - \hat{z}_{k-1}^2) \tag{4}$$

$$D_{11} = \frac{1}{3}\sum_{k=1}^{N}\overline{Q}_{11_k}(\hat{z}_k^3 - \hat{z}_{k-1}^3) \tag{5}$$

where the $\overline{Q}_{11_k}$ is the reduced-transformed stiffness of the kth lamina, $\hat{z}_k$ is its height, and N is the number of layers. The material properties are assumed not to change within a typical lamina [18,19]. The boundary conditions are given by

$$w = 0 \quad \text{and} \quad w'' = 0 \quad \text{at} \quad x = 0, 1 \tag{6}$$

$$w = 0 \quad \text{and} \quad w' = 0 \quad \text{at} \quad x = 0, 1 \tag{7}$$

for simply supported and clamped–clamped beams, respectively.

Emam and Nayfeh [17] exactly solved the nonlinear static problem of Eq. (1). The static postbuckling response corresponding to the first buckling mode is obtained as follows:

$$w_s(x) = b_s \sin\lambda x \tag{8}$$

for a simply supported beam and

$$w_s(x) = b_c(1 - \cos\lambda x) \tag{9}$$

for a clamped–clamped beam. Where b_s and b_c are two constants defined as

$$b_s = \frac{-4\Lambda}{\lambda\alpha} + \frac{2}{\sqrt{\alpha}}\sqrt{\frac{P}{\lambda^2} - 1 + \frac{\Lambda^2}{\alpha}} \tag{10}$$

$$b_c = \frac{2}{\sqrt{\alpha}}\sqrt{\frac{P}{\lambda^2} - 1} \tag{11}$$

and λ^2 is the nondimensional first critical buckling load that is equal to π^2 for simply supported beams and $4\pi^2$ for clamped–clamped beams. It is worth noting that the constant Λ vanishes for symmetric laminates, as can be noted in Eq. (2).

It is important to emphasize that Eq. (1) governs only the nonlinear free vibrations of beams in the prebuckling state. To investigate the nonlinear free vibrations of composite beams in postbuckling, one needs to introduce a dynamic disturbance to the static, buckled, equilibrium position. As such, the total transverse deformation $w(x,t)$ due to a dynamic deformation $v(x,t)$ that takes place around a static equilibrium position $w_s(x)$ can be defined as

$$w(x,t) = w_s(x) + v(x,t) \tag{12}$$

Inserting Eq. (12) into Eq. (1) yields the nondimensional equation governing large-amplitude free vibrations of composite beams in the postbuckling state. The result is

$$\begin{aligned}&\ddot{v} + v^{iv} + \lambda^2 v'' - \alpha w_s''\int_0^1 w_s' v'\, dx - \frac{1}{2}\alpha w_s''\int_0^1 v'^2\, dx \\ &- \alpha v''\int_0^1 w_s' v'\, dx - \frac{1}{2}\alpha v''\int_0^1 v'^2\, dx + \Lambda\left[w_s'(1) - w_s'(0)\right]v'' \\ &+ \Lambda[v'(1,t) - v'(0,t)](w_s'' + v'') = 0\end{aligned} \tag{13}$$

In terms of v, boundary conditions of simply supported and clamped–clamped boundary conditions are, respectively, given by

$$v = 0 \quad \text{and} \quad v'' = 0 \quad \text{at} \quad x = 0, 1 \tag{14}$$

$$v = 0 \quad \text{and} \quad v' = 0 \quad \text{at} \quad x = 0, 1 \tag{15}$$

3. Linear free vibrations

3.1. Prebuckling state

The linear free vibration problem of composite beams in the prebuckling state can be obtained from Eq. (1) by dropping the nonlinear terms. For simply supported beams, the standard procedure of solving an eigenvalue problem yields the following characteristic equation:

$$(P^2 + 4\omega_L^2)\sin k_1 \sinh k_2 = 0 \tag{16}$$

where ω_L is the linear natural frequency and k_1 and k_2 are two constants given by

$$k_{1,2} = \sqrt{\frac{1}{2}\left(\pm P + \sqrt{P^2 + 4\omega_L^2}\right)}$$

Since ω_L must be positive, Eq. (16) yields $k_1 = n\pi$, and one finds that the linear natural frequency of the first mode is given by

$$\omega_L^2 = \pi^2(\pi^2 - P) \tag{17}$$

For clamped–clamped beams, the linear natural frequency of the first mode can be obtained by solving the following transcendental equation:

$$2k_1k_2(\cos k_1 \cosh k_2 - 1) + (k_1 - k_2)(k_1 + k_2)\sin k_1 \sinh k_2 = 0 \tag{18}$$

3.2. Postbuckling state

On the other hand, the linear free vibration problem for beams in postbuckling can be obtained by dropping the nonlinear terms of Eq. (13), which is exactly solved by Nayfeh and Emam [1] for metallic beams and Emam and Nayfeh [17] for composite beams.