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Variational formulation of gradient or/and nonlocal higher-order shear elasticity beams

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ABSTRACT

This paper covers a large variety of theoretical generic beam models including some small length scale terms. Strain gradient elasticity and Eringen's nonlocal elasticity models are applied to beam mechanics including Euler–Bernoulli, Timoshenko and higher-order shear beam models. The buckling and vibration behaviour of these generalized shear beam models is investigated for pinned–pinned boundary conditions. The variational formulation of these enriched beam models is given leading to consistent variationally-based boundary conditions. The paper first starts with the axial behaviour of gradient or nonlocal elasticity bars. The beam behaviour is then analyzed using a unified framework, where the kinematics classification is presented from a generalized gradient constitutive law. It is shown that higher-order shear beam models can be classified in a common gradient elasticity Timoshenko theory, whatever the shear strain distribution assumptions over the cross section. We show the kinematics equivalence between Bickford–Reddy higher-order shear beam model and Shi–Voyiadjis higher-order shear beam model, even if both models are statically not equivalent (from the stress calculation). This equivalence is highlighted on buckling and vibrations results. The model valid for macrostructures is generalized for micro or nanostructures using some nonlocal and gradient theories to account for small scale effects, in the axial and in the bending directions. We both use the Eringen's based integral theory and the gradient theory to derive the buckling and vibration differential equations. These two theories can be connected using a generalized hybrid nonlocal law. Eringen's model is compared to a stress gradient model, whereas the gradient elasticity theory is typically a strain gradient theory. The nonlocal framework is also developed in a variational consistent framework, for bending, vibrations and buckling configurations. The nonlocality is shown to be equivalent to higher-order inertia modelling for the dynamics analysis. Buckling and vibrations solutions are presented for the nonlocal higher-order beam/column models with pinned–pinned boundary conditions. We finally analyse the main characteristics of both nonlocal and gradient theories to capture the small scale effects for micro and nanostructures. Stiffening or softening effect of gradient or nonlocal elasticity models are discussed for the buckling and the vibrations analyses.

1. Introduction

This paper is devoted to a classification of different beam models, that account for both the shear effect and the small length scale terms for small scale applications (micro or nanostructures). The analysis is restricted to a plane motion for simplicity, and only elasticity is considered for the constitutive law. Each beam model is built consistently from variational arguments in order to obtain physically and variationally-based boundary conditions. Among the different beam models available in the literature, the simplest one is the Euler–Bernoulli beam model, where the rotation function is equal to the slope angle. The constitutive law expressed at the beam level is expressed from a linear relationship between the bending moment and the in-plane curvature. This model is useful especially for thin-beam structures but has shown limitations when modelling laminates, or sandwich structures for instance when shear effect along the depth of the beam may be predominant (or for small transverse shear stiffnesses). In the Timoshenko model which has a more refined kinematics, two independent fields are assumed for the kinematics, namely the deflections and the rotations, and the constitutive law now relates both the bending moment and the shear force to the curvature and the shear strain. Even if superior to the Euler–Bernoulli beam model, the introduction of independent rotations may be not sufficient, as considered by the Timoshenko model, since this First-Order-Shear-Deformation theory implicitly assumed that any cross-section will be plane before and after deformation. Furthermore, the Timoshenko model requires shear correction factors to compensate for the error due to the constant shear stress assumption along the depth of the beam.

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Higher-order shear beam models have been developed to refine the simplified kinematics of the Timoshenko model by expressing the displacement field with polynomials of higher degree. In these refined theories, the transverse shear strain (and shear stress) is vanishing on the top and bottom edges of the beam. The higher-order shear beam models considered in this paper are the polynomial cubic-based model of Shi and Voyiadjis [1] (see more recently [2] or [3]) or the model of Bickford–Reddy ([4,5]). Some other kinematics can be also considered such as the model of Touratier [6], the model of Karama et al. [7] and the model of Mechab [8]. Challamel [9] recently shown that the higher-order shear beam model of Shi and Voyiadjis [1] can be classified as a Timoshenko gradient elasticity model. This result has been generalized recently by Challamel et al. [10] for different kind of higher-order shear beam models, where equivalent classical and higher-order shear stiffness parameters have to be calculated. When considering the buckling behaviour of these higher-order shear columns, Wang et al. [11] also considered the buckling behaviour of higher-order Bickford–Reddy's column ([4,5]). Wang et al. [11] obtained a simple and elegant buckling solution that is also used in this paper. In fact, the buckling solution of Wang et al. [11] is valid for all cubic models (including the ones of [1–5]) that are all kinematically equivalent, even if not statically equivalent. This observation was developed by Shi [2] from numerical arguments (but with different results for the stress calculation), and the equivalence is confirmed in this paper from an analytical investigation.

In addition, small length terms can be introduced through gradient elasticity or nonlocal elasticity (typically for small scale structures). Elishakoff et al. [12] presented some gradient elasticity solutions and nonlocal elasticity solutions for the Euler–Bernoulli beam theory, the Timoshenko theory and the Shi and Voyadjis higher-order shear theory, with the specific stiffening effect of the additional gradient elasticity terms (or softening effect of the nonlocal elastic model). In the present paper, the variational arguments behind each gradient or nonlocal theory are detailed, and the governing differential equations are given for each model, with the variationally-based boundary conditions. A sensitive study valid for archetypal boundary conditions indicates that smaller is stiffer with the gradient elasticity model, whereas smaller is softener with the nonlocal elasticity model. The paper starts from the axial motion analysis, and then extends the analysis to the bending case, with the Euler–Bernoulli, the Timoshenko and the higher-order shear beam modelling. Gradient elasticity and nonlocal elasticity are detailed for each theory, and a combination of both scale effects will be also presented.

2. Axial behaviour

2.1. Eringen's bar

2.1.1. Static behaviour-Second-order effects neglected

Eringen's nonlocal model [13] is defined for the uniaxial stress–strain relationship as:

$$\sigma(x) - l_c^2\sigma''(x) = E\varepsilon(x) \tag{1}$$

where σ is the uniaxial stress, ε the uniaxial strain and E the Young's modulus. A characteristic length l_c has been introduced in Eq. (1) to account for the so-called nonlocal effects. We shall discuss later the identification and the meaning of this characteristic length. As pointed out by Eringen [13], this differential equation clearly shows that the stress variable is a spatial weighted average of the strain variable where the weighting function is the Green's function of the differential system associated to the relevant boundary conditions. The axial bar constitutive law is now expressed at the cross-sectional level:

$$N(x) - l_c^2 N''(x) = EA\varepsilon(x) \quad \text{with } \varepsilon(x) = u'(x) \tag{2}$$

where N is the normal force, u is the axial displacement, and A is the cross-sectional area and $\varepsilon(x) = u'(x)$. $\varepsilon(x)$ is here the first-order strain measure. EA is the axial stiffness of the bar. The equilibrium equations are expressed thanks to the principle of virtual work:

$$\delta U[u] = \delta\pi[u] - P\delta u(L) = \int_0^L N\delta\varepsilon\, dx - P\delta u(L) = 0 \quad \text{with}$$
$$\varepsilon(x) = u'(x) \tag{3}$$

where P is a tensile axial force. The equilibrium equations by integration by parts:

$$N' = 0 \quad \text{and } [N\delta u]_0^L - P\delta u(L) = 0 \;\Rightarrow\; N(L) = P \tag{4}$$

For this problem, the normal force is constant along the bar:

$$N(x) = P \tag{5}$$

In this case, Eringen's model is equivalent to the local model as:

$$N''(x) = 0 \;\Rightarrow\; N(x) = EAu'(x) \tag{6}$$

The total potential energy of the nonlocal Eringen's bar is then written as:

$$U[u] = \pi[u] - Pu(L) = \int_0^L \frac{1}{2}N\varepsilon\, dx - Pu(L)$$
$$= \int_0^L \frac{1}{2}EAu'^2\, dx - Pu(L) \tag{7}$$

The differential equations of this problem are finally summarized below from $\delta U = 0$:

$$EAu'' = 0 \tag{8}$$

2.1.2. Eringen's bar – static behaviour-second-order effects included

When taken into account the second-order effects, the strain measure is completed with second-order terms, leading to the non-local Eringen's law with the strain:

$$N(x) - l_c^2 N''(x) = EA\varepsilon(x) \quad \text{with } \varepsilon(x) = u'(x) + \frac{1}{2}[u'(x)]^2 \tag{9}$$

The equilibrium equations are expressed thanks to the principle of virtual work:

$$\delta U[u] = \delta\pi[u] - P\delta u(L) = \int_0^L N\delta\varepsilon\, dx - P\delta u(L) = 0 \quad \text{with}$$
$$\varepsilon(x) = u'(x) + \frac{1}{2}[u'(x)]^2 \tag{10}$$

where P is a tensile axial force. The equilibrium equations are obtained by integration by parts:

$$(N(1+u'))' = 0 \quad \text{and } [N(1+u')\delta u]_0^L - P\delta u(L) = 0$$
$$\Rightarrow\; N(1+u')(L) = P \tag{11}$$

For this problem, the normal force in the deformed configuration is constant along the bar:

$$N(x)[1 + u'(x)] = P \tag{12}$$

When derivating Eq. (12) one times, one obtains:

$$N'(x) = -\frac{Pu''}{[1+u']^2} \approx -Pu'' \tag{13}$$

where the assumption that $u' \ll 1$ has been introduced. Hence, the linearized differential equations of the Eringen's problem can be summarized as:

$$N - l_c^2 N'' = EAu' \quad \text{and } N' = -Pu'' \;\Rightarrow\; (EA + P)u'' - Pl_c^2 u^{(4)} = 0 \tag{14}$$