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Exponential basis functions in the solution of laminated plates using a higher-order Zig–Zag theory

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ABSTRACT

The solution method presented in this paper is of Trefftz type which uses exponential basis functions (EBFs) to solve composite plate problems. Following its success in the solution of plates using well-known theories (*Composite Struct 2011;93:3112–9, 94:84–91 and 2012;94:2263–68*), here we aim to apply the method to higher order shear deformation theories. In this paper we demonstrate the way that one can generally evaluate the EBFs for a laminated plate using a Zig–Zag theory. We present explicit relations for three-layer sandwich plates which have a wide range of applications in structural/mechanical engineering fields. The results of our numerical experiments on the bending analysis of composites with different boundary conditions and different configurations are provided and compared with those available in the literature. For further sudies we present some results for sandwich composite plates, including those with soft cores, as new benchmarks using the Zig–Zag theory.

1. Introduction

Nowadays it is well understood that an efficient approach for the analysis of composite laminates requires a suitable theory for assuming the through-thickness variation of the state variables as well as a suitable numerical method for the analysis and approximation of the variables throughout the area representing the mid surface. Among the theories used for the through-thickness variation of the state variables, those whose unknowns are of displacement type can be divided into three main groups; equivalent single layer models (ESL), layer-wise models (LW) and Zig–Zag theories. In the ESL models, the laminate is replaced by an equivalent single-layer anisotropic plate and the assumed displacements vary continuously across the thickness of the laminate. Apart from the classical plate theory (CLPT) based on Kirchhoff's assumptions, another well-known theory in this category is the first-order shear deformation theory (FSDT) proposed by Reissner [1] and Mindlin [2]. The reader may refer to [3,4] for a more complete literature. The theory may not predict accurate results for very thick plates. In order to increase the accuracy, higher-order shear deformation theories (HSDT) have been proposed [5–8]. Among these theories, Reddy's third-order shear deformation theory (TSDT) is the most prevalent and efficient one [8]. More accurate models have then been introduced as the LW ones [9–15]. In these methods, a displacement field within each layer is prescribed. However, they are computationally expensive because the number of unknowns depends on the number of layers. It can be concluded that an ESL model with the characteristics similar to the LW models but with the number of unknowns independent of the number of layers may serve as an efficient theory. To this end different Zig–Zag theories have so far been introduced [16–19]. In these theories the continuity of the transverse shear stresses at the interface of each pair of layers is assured (see [20] for the history of the Zig–Zag methods). In this paper, as a sample of Zig–Zag methods, a higher-order Zig–Zag theory proposed in [19] is used to perform the analysis of laminated plates. In this theory, a displacement field with piece-wise linear variation is combined with another field with cubic variation. In the literature there are some studies focusing on the theory among which the readers may refer to studies in [21,22].

In the realm of numerical analysis, various methods have been developed over the past decades the most well-known of which are the finite element method (FEM) and the boundary element method (BEM). The reader may refer to [23,24] for the history of using the FEM and more general formulations for multilayered composite plates. The FEM and the BEM have a long history in the solution of problems in solid mechanics [25,26]. However, problems regarding the expense of meshing, the connectivity of the elements and the compatibility of the state variables have led to thinking of using mesh-free methods in composite plates. Among these methods, one may refer to the element free Galerkin method (EFG) [27] used for the analysis of plates in [28,29]. Furthermore, the studies by Ferreira et al. employing radial basis functions (RBFs) for the

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analysis of composite laminated plates [30,31] should be mentioned here. There is also another group of mesh-free methods using a set of points just at the boundaries. The method of fundamental solutions (MFSs) proposed by Kupradze and Aleksidze [32] is classified in the latter category. The BEM and the MFS are sometimes classified as the "Trefftz" type of methods [33,34].

In this study, we shall use a mesh-free method proposed in [35] and employed for the analysis of composite laminates in [36–40]. This method is based on the use of a series of exponential basis functions (EBFs) to satisfy the governing equation (see [41–44] for its application to other engineering problems). The coefficients are found by the imposition of the boundary conditions through a collocation approach using a discrete transformation technique introduced in [35,45,46]. Therefore, this method may also fall in the category of "Trefftz" methods and is capable of analyzing laminates with a variety of shapes and boundary conditions. In order to use the method for the analysis of composites modeled by Zig–Zag theories, we first find the corresponding EBFs. For sandwich plates with in-plane isotropy, we present the details of the procedure with explicit relations to be used as a set of library functions.

In this paper, we first overview the Zig–Zag displacement field in Section 2; afterward, we elaborate on the governing equations and the boundary conditions through a variational formulation in Section 3. In Section 4, we explain the procedure of finding the EBFs for a sampling plate problem. In Section 5, our numerical experiments are presented and the accuracy of the results in comparison with the available exact solutions is discussed. In Section 6, we summarize the conclusions made throughout the paper.

2. The displacement field

In this paper a composite plate with N orthotropic layers and total thickness of h is analyzed based on a higher-order Zig–Zag theory [19]. The displacement field, in the kth layer ($k = 1,\ldots,N$), may be written in a vector notation as

$$\begin{cases} \mathbf{u}^k = \mathbf{u}_0(x,y) - z\mathbf{w}'(x,y) + [\mathbf{F}(z)]_k \mathbf{u}_z(x,y) \\ w = w_0(x,y) \end{cases} \quad [\mathbf{F}(z)]_k = \begin{pmatrix} F_{11}(z) & F_{12}(z) \\ F_{21}(z) & F_{22}(z) \end{pmatrix}_k, \tag{1}$$

where $\mathbf{u}^k = [u, v]_k^T$, $\mathbf{u}_0 = [u_0, v_0]^T$, $\mathbf{w}' = [\partial w_0/\partial x, \partial w_0/\partial y]^T$ and $\mathbf{u}_z = [u_z, v_z]^T$, so that (u_0, v_0, w_0) are the mid-plane displacements and (u_z, v_z) are the Zig–Zag displacement terms. According to the assumed displacement field, the unknown vector to be found is $[u_0, v_0, w_0, u_z, v_z]^T$. The Zig–Zag part of the displacements in (1), i.e. $[\mathbf{F}(z)]_k$, may be expressed as

$$[\mathbf{F}(z)]_k = \left\{ \mathbf{H}_1^k + z^2\mathbf{H}_2 + z^3\mathbf{I}_{2\times 2} \right\}, \tag{2}$$

where $\mathbf{I}$ is an identity matrix and $\mathbf{H}_1^k$ and $\mathbf{H}_2$ are obtained by the procedure given in Appendix A.

3. The governing equations and boundary conditions

In this section, we proceed to find the governing equations through a variational approach (see also [24] for further details in more general cases). In a static state, the variation of the total potential of the composite plate is expressed as

$$\delta\Pi = \delta U + \delta V = 0, \tag{3}$$

in which

$$\delta U = \int_{\Omega_0} \left\{ \int_{-\frac{h}{2}}^{\frac{h}{2}} [\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + 2\sigma_{xy}\delta\varepsilon_{xy} + 2\sigma_{xz}\delta\varepsilon_{xz} + 2\sigma_{yz}\delta\varepsilon_{yz}]dz \right\} d\Omega_0, \tag{4}$$

In the above relations, Ω_0 denotes the domain of the mid-plane defined in x and y. Also σ and ε represent the components of the stress and strain tensors, respectively. The second term in (3), δV, pertains to the variation of the potential of the applied loads (we shall refer to this part when boundary conditions are to be derived).

By substituting the strain–displacement relations, as well as the constitutive relations, in (4) and performing integral by parts, a variational expression in terms of δu_0, δv_0, δw_0, δu_z and δv_z is resulted. The expressions conjugate with the displacement variations represent the governing differential equation as

$$\mathbf{L}\tilde{\mathbf{u}} = \mathbf{q} \quad \text{in} \quad \Omega_0 \tag{5}$$

where $\mathbf{L}$ is an operator matrix, not elaborated here for the sake of brevity, $\tilde{\mathbf{u}} = [u_0, v_0, w_0, u_z, v_z]^T$ is the unknown vector to be found, and $\mathbf{q}$ is a vector containing the loading components defined as $\mathbf{q} = [0, 0, q_z, 0, 0]^T$. It is important to note that when $\mathbf{L}$ is operated on a displacement field of a plate with symmetric layer sequence, some of its elements vanish. Therefore the operator matrix may be condensed to $\mathbf{L}^{sub}_{(3\times 3)}$ as

$$\mathbf{L}^{sub}\tilde{\mathbf{u}} = \mathbf{q}, \tag{6}$$

where $\tilde{\mathbf{u}}$ is now defined as $[w_0, u_z, v_z]^T$, and $\mathbf{q}$ is the corresponding loading vector defined as $[q_z, 0, 0]^T$.

3.1. Boundary conditions

For derivation of the boundary conditions, we focus on δV in (3) with a general expression as

$$\delta V = \delta V_{\Omega_0} + \delta V_{\partial\Omega_0}. \tag{7}$$

In the above relation $\partial\Omega_0$ represents the boundary of Ω_0. Also δV_{Ω_0} denotes part of δV pertaining to the transverse load while $\delta V_{\partial\Omega_0}$ denotes part of δV affected by the boundary conditions. Regardless of the type of the boundary conditions, δV_{Ω_0} may be written as

$$\delta V_{\partial\Omega_0} = -\int_{s\equiv\partial\Omega_0} \left\{ \int_{-\frac{h}{2}}^{\frac{h}{2}} [\hat{\sigma}_{nn}\delta u_n + \hat{\sigma}_{ns}\delta u_s + \hat{\sigma}_{nz}\delta w]dz \right\} ds, \tag{8}$$

in which Neumann and Dirichlet conditions appear as a set of conjugate pairs. In the above relation $\hat{\sigma}_{nn}$, $\hat{\sigma}_{ns}$ and $\hat{\sigma}_{nz}$ are the stress components on $\partial\Omega_0$ in the directions of n and s, normal and tangent to the boundary. For Neumann conditions the stresses are prescribed but at this stage there is no need to distinguish them from Dirichlet ones. In (8) δu_n and δu_s denote the variation of the components of $\mathbf{u}$ in (1) evaluated in the directions of n and s. By defining

$$\delta\mathbf{u}_n = [\,\delta u_n \quad \delta u_s\,]^T, \quad \delta\mathbf{u} = [\,\delta u \quad \delta v\,]^T, \tag{9}$$

one may write

$$\delta\mathbf{u}_n = \mathbf{n}\delta\mathbf{u}, \quad \mathbf{n} = \begin{bmatrix} n_x & n_y \\ -n_y & n_x \end{bmatrix}, \tag{10}$$

where n_x and n_y are the components of the unit vector normal to the boundary. In view of (1), for kth layer we have

$$\begin{aligned} &\delta\mathbf{u}_n^k = \delta\mathbf{u}_{0n} - z\delta\mathbf{w}_n' + [\mathbf{F}(z)]_k\delta\mathbf{u}_{nz}, \\ &\delta\mathbf{w}' = [\partial(\delta w_0)/\partial n, \partial(\delta w_0)/\partial s]^T. \end{aligned} \tag{11}$$

By using (11) in (8), and evaluating the integrals in z direction, we arrive at

$$\begin{aligned} \delta V_{\partial\Omega_0} = -\int_{s\equiv\partial\Omega_0} &\{\delta u_{0n}N_{nn} + \delta u_{0s}N_{ns} + \delta u_{zn}M^*_{nn} + \delta u_{zs}M^*_{ns} \\ &+ [\partial(\delta w_0)/\partial n]M_{nn} - [\partial(\delta w_0)/\partial s]M_{ns} + \delta w_0 Q_n\}ds \end{aligned} \tag{12}$$

The signs in the above relation depend on the positive direction of each component of the conjugate pairs.