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Solution methods of exact solutions for free vibration of rectangular orthotropic thin plates with classical boundary conditions

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ABSTRACT

The exact solutions for free vibrations of orthotropic rectangular thin plates are presented in an elegant way using separation of variables method. And the exact solutions of three configurations (G-G-C-C, SS-G-C-C and C-C-C-G) are solved for the first time. In separation of variables method, the general formulation of the natural mode function and the relations among two spatial eigenvalues and a temporal eigenvalue are directly obtained from the characteristic differential equation, and the coefficients of exact mode functions and two exact eigenvalue equations are determined by two pairs of opposite edge conditions. It was regarded before 2009 that there were not exact solutions for rectangular thin plates with at least two adjacent clamped edges, and the others to be arbitrary combinations of clamped (C), simply supported (SS) and guided (G) conditions. For these configurations, the exact eigensolutions are solved successfully and the first ten exact frequencies are tabulated for a few of aspect ratios.

1. Introduction

The vibration problem of rectangular plates, although more than some two hundred years old in its research account, continues to be of considerable research interest, for the reason that rectangular plates are basic structural elements and hence practical applications may involve enormous parametric variations as loading, materials, aspect ratio and support conditions, etc. [1]. Another reason is that there are some configurations for which we have not solved the exact solutions until now. For example, it was known before 2009 that there were not exact solutions for the rectangular thin plate with two adjacent clamp edges or two adjacent free edges.

Linear free vibration problems of rectangular thin plates fall into three distinct categories: (a) plates with all edges simply supported or/and guided (altogether six configurations); (b) plates with only a pair of opposite edges simply supported or/and guided (altogether 21 configurations); (c) plates which do not fall into any of the above categories (altogether 28 configurations). Problems of the first and second categories are amenable to straightforward rigorous analysis in terms of the well-known Navier and Levy solutions. However, owing to coupled multiple differential equations of high order, it was believed before 2009 that there were not exact solutions for the free vibration of the third category problems [1–3] or difficult to obtain the exact solutions [4]. For this reason many efforts were devoted to develop approximate methods [5–7 for example]. But in present work, the exact solutions of three configurations (G-G-C-C, C-C-C-G and SS-G-C-C) of the third category problems are found using the separation of variables method [8,9]. It is noteworthy that, in 2009, the author has obtained the exact solutions of the three configurations (SS-SS-C-C, SS-C-C-C and C-C-C-C) of the third category problems [8,9]. Comparing with Ref. [9], the novelties of present paper are as follows:

(1) The exact solutions are presented for rectangular orthotropic thin plates with G-G-C-C, C-C-C-G and SS-G-C-C configurations.
(2) The first ten frequencies are tabulated for the rectangular isotropic and orthotropic thin plates with SS-SS-C-C, SS-C-C-C, C-C-C-C G-G-C-C, C-C-C-G and SS-G-C-C boundaries.
(3) All possible exact solutions for rectangular thin plates are presented in a compact and elegant way.

One of the intents of this paper is to give the new exact solutions of free vibration for rectangular thin plates, and the other is to present all possible exact frequency equations and natural mode functions for rectangular thin plates in an elegant method. And the first ten frequencies of the six configurations of isotropic and orthotropic plates with two adjacent clamped edges and the others to be arbitrary combinations of C, SS and G conditions are tabulated for a range of the plate aspect ratios for the first time. A few words are given for the numerical solution methods of transcendental eigenvalue equations.

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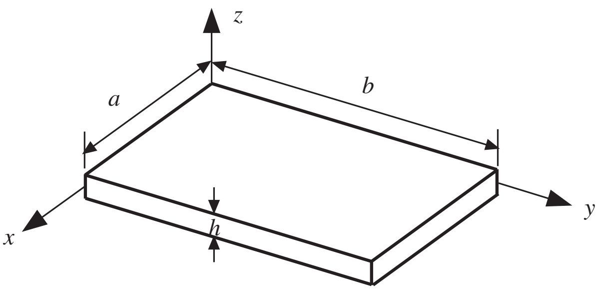

Fig. 1. A rectangular plate and coordinates.

2. Exact solutions in separation of variables form

There are four independent elastic constants E_1, E_2, G_{12} and υ_{12} for orthotropic materials having two perpendicular planes of symmetry. The characteristic equation of orthotropic rectangular thin plate, see Fig. 1, has the form

$$D_1\frac{\partial^4 W}{\partial X^4}+2D_3\theta^2\frac{\partial^4 W}{\partial X^2\partial Y^2}+D_2\theta^4\frac{\partial^4 W}{\partial Y^4}-a^4\rho h\omega^2 W=0 \tag{1}$$

where dimensionless coordinates $X = x/a$, $Y = y/b$ and aspect ratio $\theta = a/b$ are introduced, $W(X, Y)$ is the natural mode function and the orthotropic bending stiffness parameters are

$$D_1=\frac{E_1h^3}{12(1-\upsilon_{12}\upsilon_{21})},\quad D_2=\frac{E_2h^3}{12(1-\upsilon_{12}\upsilon_{21})},\quad D_{66}=\frac{G_{12}h^3}{12} \tag{2}$$
$$D_{12}=\upsilon_{12}D_2=\upsilon_{21}D_1,\quad D_3=D_{12}+2D_{66}$$

where h and ρ are the thickness of plate and the volume mass density of the material. The solutions of Eq. (1) depend basically on the edge boundary conditions. The classical boundary conditions employed for the analysis of plates are simply supported, clamped, free (F) and guided conditions, and with all possible combinations of these conditions at the four edges, 55 rectangular plate configurations are possible [1]. Of these, it was believed that only 27 configurations had exact solutions and an explicit form of frequency equation exists in each of these cases with a pair of opposite edges simply supported or/and guided. In this work a separation of variables solution of Eq. (1) is assumed

$$W(X,Y)=\phi(X)\psi(Y) \tag{3}$$

where ϕ and ψ are respective x-direction and y-direction mode functions or eigenfunctions, and they have the forms as

$$\phi(X)=Ae^{\mu X},\quad \psi(Y)=Be^{\lambda Y} \tag{4}$$

where the variables μ and λ are the eigenvalues of x and y directions, respectively. In classical separation of variables method, one of μ and λ and its corresponding eigenfunction is given or assumed for the pair of opposite simply supported or/and guided edges, then the plate problems become to be beamlike problems. On the contrary, these two spatial eigenvalues and frequency are all unknowns and solved simultaneously in our separation of variables method [8,9], in which the general formulation of the natural mode function $W(X, Y)$ and the relationship among the two spatial eigenvalues and the temporal eigenvalue are solved from the characteristic differential equation, and the coefficients of the exact mode function $W(X, Y)$ and the exact eigenvalue equations are determined via boundary conditions of four edges.Substituting of expressions (3) and (4) into Eq. (1), we have

Table 1
Y-Direction eigensolutions.

B.C.	Eigenvalue equations	Eigenfunctions
SS-SS G-G	$\sin\alpha_2=0$	$\psi(Y)=\sin\alpha_2 Y$ $\psi(Y)=\cos\alpha_2 Y$
C-C	$\frac{1-\cos\alpha_2\cosh\beta_2}{\sin\alpha_2\sinh\beta_2}=\frac{\alpha_2^2-\beta_2^2}{2\alpha_2\beta_2}$	$\psi(Y)=-\cos\alpha_2 Y+\frac{\beta_2}{\alpha_2}k_{1(\alpha_2,\beta_2)}\sin\alpha_2 Y+\cosh\beta_2 Y-k_{1(\alpha_2,\beta_2)}\sinh\beta_2 Y$
SS-G G-SS	$\cos\alpha_2=0$	$\psi(Y)=\sin\alpha_2 Y$ $\psi(Y)=\cos\alpha_2 Y$
SS-C C-SS	$\beta_2\tan\alpha_2=\alpha_2\tanh\beta_2$	$\psi(Y)=\sin\alpha_2 Y-\frac{\sin\alpha_2}{\sinh\beta_2}\sinh\beta_2 Y$ $\psi(Y)=-\cos\alpha_2 Y+\frac{\beta_2}{\alpha_2}k_{1(\alpha_2,\beta_2)}\sin\alpha_2 Y+\cosh\beta_2 Y-k_{1(\alpha_2,\beta_2)}\sinh\beta_2 Y$
G-C C-G	$\alpha_2\tan\alpha_2+\beta_2\tanh\beta_2=0$	$\psi(Y)=\cos\alpha_2 Y-\frac{\cos\alpha_2}{\cosh\beta_2}\cosh\beta_2 Y$ $\psi(Y)=-\cos\alpha_2 Y-\frac{\beta_2}{\alpha_2}k_{2(\alpha_2,\beta_2)}\sin\alpha_2 Y+\cosh\beta_2 Y+k_{2(\alpha_2,\beta_2)}\sinh\beta_2 Y$
$k_{1(\alpha_2,\beta_2)}=\frac{\alpha_2(\cos\alpha_2-\cosh\beta_2)}{\beta_2\sin\alpha_2-\alpha_2\sinh\beta_2},k_{2(\alpha_2,\beta_2)}=\frac{\alpha_2\sin\alpha_2+\beta_2\sinh\beta_2}{\beta_2(\cos\alpha_2-\cosh\beta_2)}$		

Table 2
Eigensolutions of plate with a combination of SS and G conditions for the pair of edges $Y = 0, 1$ and involving a free edge for remaining two edges.

B.C.	Eigenvalue equations	Eigenfunctions
SS-F F-SS	$c_1d_2\tan\alpha_1=c_2d_1\tanh\beta_1$	$\phi(X)=\sin\alpha_1 X-\frac{c_1\sin\alpha_1}{c_2\sinh\beta_1}\sinh\beta_1 X$ $\phi(X)=-\frac{c_2}{c_1}\cos\alpha_1 X-\frac{d_2}{d_1}k_{1(\alpha_1,\beta_1)}\sin\alpha_1 X+\cosh\beta_1 X+k_{1(\alpha_1,\beta_1)}\sinh\beta_1 X$
C-F F-C	$(c_2d_1\alpha_1-c_1d_2\beta_1)\sin\alpha_1\sinh\beta_1+(c_1d_2\alpha_1+c_2d_1\beta_1)\cos\alpha_1\cosh\beta_1-(c_1d_1\beta_1+c_2d_2\alpha_1)=0$	$\phi(X)=-\cos\alpha_1 X-\frac{\beta_1}{\alpha_1}k_{2(\alpha_1,\beta_1)}\sin\alpha_1 X+\cosh\beta_1 X+k_{2(\alpha_1,\beta_1)}\sinh\beta_1 X$ $\phi(X)=-\frac{c_2}{c_1}\cos\alpha_1 X-\frac{d_2}{d_1}k_{1(\alpha_1,\beta_1)}\sin\alpha_1 X+\cosh\beta_1 X+k_{1(\alpha_1,\beta_1)}\sinh\beta_1 X$
G-F F-G	$c_1d_2\tanh\beta_1+c_2d_1\tan\alpha_1=0$	$\phi(X)=\cos\alpha_1 X-\frac{c_1\cos\alpha_1}{c_2\cosh\beta_1}\cosh\beta_1 X$ $\phi(X)=-\frac{c_2}{c_1}\cos\alpha_1 X-\frac{d_2}{d_1}k_{3(\alpha_1,\beta_1)}\sin\alpha_1 X+\cosh\beta_1 X+k_{3(\alpha_1,\beta_1)}\sinh\beta_1 X$
F-F	$\frac{1-\cos\alpha_1\cosh\beta_1}{\sin\alpha_1\sinh\beta_1}=\frac{c_2^2d_1^2-c_1^2d_2^2}{2c_1d_1c_2d_2}$	$\phi(X)=-\frac{c_2}{c_1}\cos\alpha_1 X-\frac{d_2}{d_1}k_{4(\alpha_1,\beta_1)}\sin\alpha_1 X+\cosh\beta_1 X+k_{4(\alpha_1,\beta_1)}\sinh\beta_1 X$
$k_{1(\alpha_1,\beta_1)}=-\frac{d_1(c_2\cos\alpha_1-c_1\cosh\beta_1)}{c_1(d_2\sin\alpha_1-d_1\sinh\beta_1)},k_{2(\alpha_1,\beta_1)}=-\frac{\alpha_1(c_1\cos\alpha_1-c_2\cosh\beta_1)}{c_1\beta_1\sin\alpha_1-c_2\alpha_1\sinh\beta_1}$ $k_{3(\alpha_1,\beta_1)}=\frac{d_1(c_2\alpha_1\sin\alpha_1+c_1\beta_1\sinh\beta_1)}{c_1(d_2\alpha_1\cos\alpha_1-d_1\beta_1\cosh\beta_1)},k_{4(\alpha_1,\beta_1)}=-\frac{c_2d_1(\cos\alpha_1-\cosh\beta_1)}{c_1d_2\sin\alpha_1-c_2d_1\sinh\beta_1}$		