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A simple quasi-3D sinusoidal shear deformation theory for functionally graded plates

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ABSTRACT

This paper presents a simple quasi-3D theory for the bending analysis of functionally graded plates. This theory accounts for both shear deformation and thickness stretching effects by a sinusoidal variation of all displacements through the thickness. By dividing the transverse displacement into bending and shear parts, the number of unknowns and governing equations of the present theory is reduced, and hence, makes it simple to use. The governing equations and boundary conditions are derived using the principle of virtual displacements. Analytical solutions are obtained for simply supported plates. The accuracy of the present theory is verified by comparing the obtained results with 3D and quasi-3D solutions and those predicted by higher-order shear deformation theories. The comparison studies show that the obtained results are not only more accurate than those obtained by higher-order shear deformation theories, but also comparable with those predicted by quasi-3D theories with a greater number of unknowns.

1. Introduction

Functionally graded materials (FGMs) are a class of composites that have continuous variation of material properties from one surface to another, thus eliminating the stress concentration found in laminated composites. A typical FGM is made from a mixture of two material phases, for example, a ceramic and a metal. The reason for the increasing use of FGMs in a variety of aerospace, automotive, civil, and mechanical engineering structures is that their material properties can be tailored to different applications and working environments. The increase of FGM applications requires the development of accurate theories to predict their responses. It should be noted that the classical plate theory (CPT), which is based on the Kirchhoff hypothesis, is suitable for thin plates, but inadequate for thick plates or plates made of advanced composites like FGMs. The first-order shear deformation theory (FSDT) [1–3] accounts for the shear deformation effect by the way of linear variation of in-plane displacements through the thickness. Thus, a shear correction factor is required to compensate for the difference between the actual and assumed constant stress states [4,5]. The higher-order shear deformation theory (HSDT) [6–23] accounts for the shear deformation effect by the way of a higher-order variation of in-plane displacements through the thickness, and hence, the shear correction factor is not required. It should be noted that the above-mentioned 2-D plate theories (i.e., CPT, FSDT, and HSDT) discard the thickness stretching effect (i.e., $\varepsilon_z = 0$) due to assuming a constant transverse displacement through the thickness. This assumption is appropriate for thin or moderately thick functionally graded (FG) plates, but is inadequate for thick FG plates [24]. The importance of the thickness stretching effect in FG plates has been pointed out in the work of Carrera et al. [25]. This effect plays a significant role in moderately thick and thick FG plates and should be taken into consideration.

Quasi-3D theories are HSDTs with higher-order variations through the thickness for the transverse displacement. In general, quasi-3D theories can be implemented using the unified formulation initially proposed by Carrera [26–28] and recently extended by Demasi [29–34]. More detailed information and applications of the unified formulation can be found in the recent books by Carrera et al. [35,36]. Since the quasi-3D theory accounts for a higher-order variation of both in-plane and transverse displacements through the thickness, both the shear deformation effect and the thickness stretching effect are considered. Many quasi-3D theories have been proposed in the literature [37–44]. These theories are cumbersome and computationally expensive due to having a host of unknowns (e.g., theories by Talha and Singh [39] with thirteen unknowns; Chen et al. [38] and Reddy [41] with eleven unknowns; and Ferreira et al. [40] and Neves et al. [42–44] with nine unknowns). Although some well-known quasi-3D theories developed by Zenkour [45] and recently by Mantari and

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Guedes Soares [46,47] have six unknowns, they are still more complicated than the FSDT. Thus, developing a simple quasi-3D theory is necessary.

This paper aims to develop a simple quasi-3D theory with only five unknowns. The displacement field is chosen based on a sinusoidal variation of in-plane and transverse displacements through the thickness. The partition of the transverse displacement into the bending and shear parts leads to a reduction of the number of unknowns, and subsequently, makes the new theory simple to use. Governing equations and boundary conditions are derived using the principle of virtual displacements. Analytical solutions for deflections and stresses are obtained for a simply supported rectangular plate. Numerical examples are presented to verify the validity of the present theory.

2. Theoretical formulation

2.1. Kinematics

The aim of this paper is to develop a simple quasi-3D theory in which in-plane and transverse displacements are expanded as a sinusoidal variation through the thickness. The advantages of the sinusoidal functions over the polynomial functions are that they are simple and accurate, and the stress-free boundary conditions on the top and bottom surfaces of the plate can be guaranteed [48]. In fact, the use of sinusoidal functions was first proposed by Levy [49] and assessed by Stein [50], and later extensively used by Touratier [51] and Zenkour [9]. In this study, further simplifying assumptions are made to the quasi-3D theory of Zenkour [45] so that the number of unknowns is reduced by one. According to Zenkour [45], the displacement field of the quasi-3D sinusoidal theory is given by

$$\begin{aligned} u_1(x,y,z) &= u(x,y) - z\frac{\partial w}{\partial x} + \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)\varphi_x \\ u_2(x,y,z) &= v(x,y) - z\frac{\partial w}{\partial y} + \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)\varphi_y \\ u_3(x,y,z) &= w(x,y) + \cos\left(\frac{\pi z}{h}\right)\varphi_z(x,y) \end{aligned} \tag{1}$$

where u, v, w, φ_x, φ_y and φ_z are six unknown displacement functions of the midplane of the plate; and h is the thickness of the plate. Further assumptions are given by

$$w(x,y) = w_b(x,y) + w_s(x,y), \quad \varphi_x = \frac{\partial w_s}{\partial x} \quad \text{and } \varphi_y = \frac{\partial w_s}{\partial y} \tag{2}$$

Substituting Eq. (2) into Eq. (1), the displacement field of the proposed theory takes the simpler form as follows:

$$\begin{aligned} u_1(x,y,z) &= u(x,y) - z\frac{\partial w_b}{\partial x} - \left[z - \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)\right]\frac{\partial w_s}{\partial x} \\ u_2(x,y,z) &= v(x,y) - z\frac{\partial w_b}{\partial y} - \left[z - \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)\right]\frac{\partial w_s}{\partial y} \\ u_3(x,y,z) &= w_b(x,y) + w_s(x,y) + \cos\left(\frac{\pi z}{h}\right)\varphi_z(x,y) \end{aligned} \tag{3}$$

Clearly, the displacement field in Eq. (3) contains only five unknowns (u, v, w_b, w_s, φ_z). The linear strains associated with the displacement field in Eq. (3) are:

$$\varepsilon_x = \frac{\partial u}{\partial x} - z\frac{\partial^2 w_b}{\partial x^2} - f(z)\frac{\partial^2 w_s}{\partial x^2} \tag{4a}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} - z\frac{\partial^2 w_b}{\partial y^2} - f(z)\frac{\partial^2 w_s}{\partial y^2} \tag{4b}$$

$$\varepsilon_z = g'(z)\varphi_z \tag{4c}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z\frac{\partial^2 w_b}{\partial x \partial y} - 2f(z)\frac{\partial^2 w_s}{\partial x \partial y} \tag{4d}$$

$$\gamma_{xz} = g(z)\left(\frac{\partial w_s}{\partial x} + \frac{\partial \varphi_z}{\partial x}\right) \tag{4e}$$

$$\gamma_{yz} = g(z)\left(\frac{\partial w_s}{\partial y} + \frac{\partial \varphi_z}{\partial y}\right) \tag{4f}$$

where $f(z) = z - \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)$ and $g(z) = 1 - f'(z) = \cos\left(\frac{\pi z}{h}\right)$. It can be seen from Eqs. (4e) and (4f) that the transverse shear strains (γ_{xz}, γ_{yz}) are equal to zero at the top ($z = h/2$) and bottom ($z = -h/2$) surfaces of the plate, thus satisfying the zero transverse shear stress conditions.

2.2. Constitutive equations

Consider FG plates made from a mixture of two material phases, for example, a metal and a ceramic as show in Fig. 1. Poisson's ratio ν is assumed to be constant, while Young's modulus $E(z)$ varies continuously through the plate thickness by either an exponential or a polynomial material law. According to the exponential material law, the effective Young's modulus $E(z)$ is estimated as [45]

$$E(z) = E_0\bar{f}(z), \quad \bar{f}(z) = e^{p(0.5+z/h)} \tag{5a}$$

where $E_m = E_0$ and $E_c = E_0e^p$ denote the Young's modulus of the bottom (as metal) and top (as ceramic) surfaces of the FG plate, respectively; E_0 is Young's modulus of the homogeneous plate; and p is a parameter that indicates the material variation through the plate thickness and takes values greater than or equal to zero. According to the polynomial material law, the effective Young's modulus $E(z)$ is estimated as [9]

$$E(z) = E_m + (E_c - E_m)(0.5 + z/h)^p \tag{5b}$$

The constitutive relations of a FG plate can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \tag{6}$$

where C_{ij} are the three-dimensional elastic constants given by

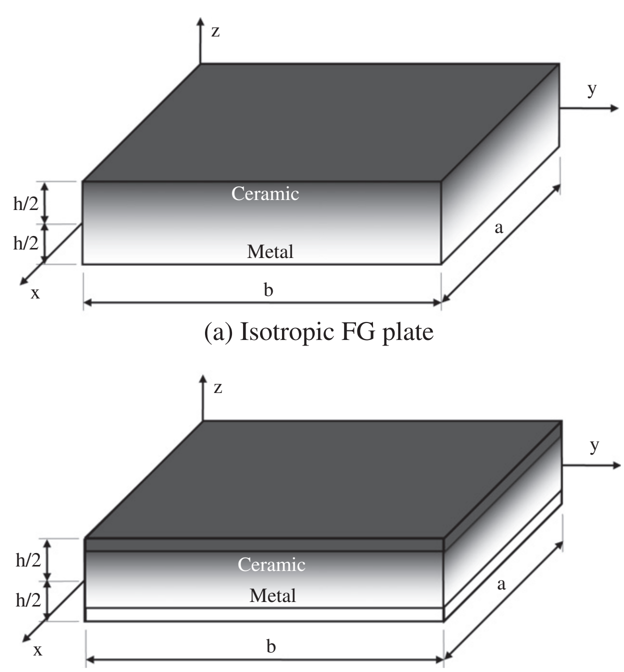

Fig. 1. Geometry and coordinates of FG plates.