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Partially Zig-Zag Advanced Higher Order Shear Deformation Theories Based on the Generalized Unified Formulation

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A B S T R A C T

The Generalized Unified Formulation (GUF) is a multi-theory and a multi-fidelity architecture for the generation of a virtually infinite class of Advanced Higher Order Shear Deformation Theories (AHSDT) or Zig-Zag theories or Layer-Wise (LW) theories with any order of expansion for each of the primary variables. This work will present, for the first time in the literature, an extension of GUF to address problems in which every single variable can have either an Equivalent Single Layer (ESL) or a Zig-Zag-enhanced ESL description [Partially Zig-Zag Advanced Higher Order Shear Deformation Theories (PZZAHSDT)]. Applications to the case of thick sandwich structures are presented: starting from a baseline fourth-order AHSDT which also includes the transverse strain effects, all the possible types of PZZAHSDT are generated and compared with the baseline and with a fourth-order fully Zig-Zag theories.

1. Introduction

The analysis of composite and sandwich structures requires a model that can take into account the strong anisotropy along the thickness and the shear deformation effects. Classical Plate Theory [33] is in general not adequate especially for moderately thick or thick plates. The relaxation of the Kirchhoff assumptions led to the formulation of the First Order Shear Deformation Theory (FSDT) [44,35,32] in which a generic planar cross-section initially orthogonal to the mid-plane of the plate after the deformation takes place is still planar but no longer perpendicular to the mid-surface of the plate. An initially planar cross-section can be allowed to deform in a generic shape with the inclusion (usually for the in-plane displacements only) of higher order terms in the axiomatic expansion along the thickness. These approaches are the so called Higher Order Shear Deformation Theories (HSDT) [54,7,30,56]. Transverse strain effects may be added by including higher order terms in the thickness expansion for the out-of-plane displacement u_z. The resulting theories are here indicated as Advanced Higher Order Shear Deformation Theories (AHSDT). However, as will be demonstrated in detail in this paper, due to both the interlaminar equilibrium of the transverse stresses and anisotropy of the mechanical properties along the thickness, all the displacements and stresses (with the only exception of the transverse normal stress) present a discontinuity of the first spatial derivative with respect to the thickness coordinate z. The discontinuity of the displacement variables is what people refer to "Zig-Zag form of the displacements" [6,1,31,53]. Based on this physical evidence it has been proposed by many researchers to include the Zig-Zag form of the displacements a priori with a model that is still a computationally inexpensive formulation but has a significant improvement of the results due to a better physical representation of the real deformation of the anisotropic composite structure. Following the historical reconstruction attempted in Ref. [13] on this subject, the Zig-Zag theories can be subdivided into three major groups:

- Lekhnitskii Multilayered Theory (LMT)
- Ambartsumian Multilayered Theory (AMT)
- Reissner Multilayered Theory (RMT)

In the Lekhnitskii Multilayered Theory [34] (originally formulated for multilayered beams) the Zig-Zag form of the displacements and continuity of the transverse stresses were enforced. LMT was extended to the case of plates by Ren [48,47].

In the Ambartsumian Multilayered Theory [5,4,2,3] (formulated for both plates and shells) an interlaminar continuous transverse shear stress field is a priori enforced. The displacement fields present a discontinuity of the first derivatives in the thickness direction (Zig-Zag form). Later the effects of transverse normal strain/stress were also included [51,52,38–40]. Whitney [55] applied AMT to anisotropic and non-symmetrical plates. Later [41] Rath and Das extended Whitney's work to shells and dynamic problems. Other authors such as Yu [57], Chou and Carleone [18] and Di Sciuva [29] worked on similar (but *less general*) approaches. Cho and Parmerter refined [17] these alternative approaches and obtained a Zig-Zag formulation which was equivalent to AMT.

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In the Reissner Multilayered Theory the transverse stresses are primary unknowns as well as the displacement variables [45,46]. The variational statement is Reissner's Mixed Variational Theorem. Murakami [36] proposed to take into account the Zig-Zag effects by *enhancing* the corresponding displacement variable with a Zig-Zag function denoted here as Murakami's Zig-Zag Function (MZZF). Applications of the concept of enhancing the displacement field with MZZF were presented in several works [15,19,21,26,10,9,8,50] in the last few years. The main advantage of Zig-Zag approaches is in the significant improvement of the accuracy with little increment of the computational cost with respect to the inexpensive (but often inaccurate) classical formulations.

For more challenging problems (for example a sandwich structure with a very high Face-to-Core Stiffness Ratio) a LW [16,37,42,49,43,12,11] approach is necessary and represents a valuable alternative to the computationally very expensive Finite Element discretizations based on solid elements. All of these LW and ESL approaches can be unified with the adoption of the so called *Compact Notations* (CN). Example of Compact Notations are Carrera's Unified Formulation (CUF) [14] and its generalization represented by the Generalized Unified Formulation (GUF) [22–28]. The main idea behind GUF is the writing of each displacement variable (or/and stress variable in the case of mixed formulations) independently from the other unknowns. With this approach, any combination of orders can be achieved. For example, an AHSDT with cubic thickness expansion for the in-plane displacements and a parabolic expansion for the out-of-plane displacement can be represented as well as a LW theory with parabolic expansion of the in-plane displacement variables and linear expansion for the transverse displacement u_z.

1.1. What are the new contributions of this work

Up to now the Generalized Unified Formulation could handle any combination of orders for the displacement unknowns. However, the type of description was the same for all the displacements (e.g., LW or ESL description for all the variables). With this work and for the first time, GUF is further generalized. In particular, it is presented the case in which *some* variables can be described in an ESL form and others will be enhanced with MZZF. This work first discusses the physical and mathematical justifications of the need of Zig-Zag form of the displacements and stresses (with the only exception represented by the transverse normal stress σ_{zz} as will be discussed later) and then presents the main theoretical aspects and results.

2. Zig-Zag form of the displacements

A mathematical explanation of the Zig-Zag form of the displacements is discussed. Classical Form of Hooke's Law (CFHL) in plate coordinates [43,23] is:

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\\ \sigma_{zx}\\ \sigma_{zy}\\ \sigma_{zz}\end{bmatrix}=\begin{bmatrix}\widetilde{C}_{11} & \widetilde{C}_{12} & \widetilde{C}_{16} & 0 & 0 & \widetilde{C}_{13}\\ \widetilde{C}_{12} & \widetilde{C}_{22} & \widetilde{C}_{26} & 0 & 0 & \widetilde{C}_{23}\\ \widetilde{C}_{16} & \widetilde{C}_{26} & \widetilde{C}_{66} & 0 & 0 & \widetilde{C}_{36}\\ 0 & 0 & 0 & \widetilde{C}_{55} & \widetilde{C}_{45} & 0\\ 0 & 0 & 0 & \widetilde{C}_{45} & \widetilde{C}_{44} & 0\\ \widetilde{C}_{13} & \widetilde{C}_{23} & \widetilde{C}_{36} & 0 & 0 & \widetilde{C}_{33}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\\ \gamma_{zx}\\ \gamma_{zy}\\ \varepsilon_{zz}\end{bmatrix} \tag{1}$$

where the explicit expressions for the coefficients can be found in Ref. [43]. With the exception of the Zig-Zag term (in the Zig-Zag term k represents an actual exponent as will be specifically seen later), whenever superscript/subscript k is present it means that the corresponding quantity is referred to layer k. From CFHL and the geometric relations, which relate the strains to the displacements, the transverse shear stress σ_{zx} evaluated at the *top surface* ($z = z_{\text{top}_k}$) of layer k can be written as

$$\begin{aligned}[\sigma_{zx}]_{z=z_{\text{top}_k}} &\equiv \sigma_{zx}^{k\,t} = \widetilde{C}_{55}^{k}\gamma_{zx}^{k\,t} + \widetilde{C}_{45}^{k}\gamma_{zy}^{k\,t}\\ &= \widetilde{C}_{55}^{k}\left(\frac{\partial u_z^{k\,t}}{\partial x} + \frac{\partial u_x^{k\,t}}{\partial z}\right) + \widetilde{C}_{45}^{k}\left(\frac{\partial u_z^{k\,t}}{\partial y} + \frac{\partial u_y^{k\,t}}{\partial z}\right)\end{aligned} \tag{2}$$

the superscript t is used to highlight the fact that the *top* surface of layer k is considered. The same relationship written at the *bottom surface* ($z = z_{\text{bot}_{(k+1)}}$) of layer $k+1$ can be written as

$$\begin{aligned}[\sigma_{zx}]_{z=z_{\text{bot}_{(k+1)}}} &\equiv \sigma_{zx}^{(k+1)\,b}\\ &= \widetilde{C}_{55}^{(k+1)}\left(\frac{\partial u_z^{(k+1)\,b}}{\partial x} + \frac{\partial u_x^{(k+1)\,b}}{\partial z}\right)\\ &\quad + \widetilde{C}_{45}^{(k+1)}\left(\frac{\partial u_z^{(k+1)\,b}}{\partial y} + \frac{\partial u_y^{(k+1)\,b}}{\partial z}\right)\end{aligned} \tag{3}$$

the superscript b is used to highlight the fact that the *bottom* surface of layer $k+1$ is considered. For the transverse shear stress σ_{zy} similar finding can be obtained from CFHL (see Eq. (1)):

$$\begin{aligned}\sigma_{zy}^{(k+1)\,b} &= \widetilde{C}_{45}^{(k+1)}\left(\frac{\partial u_z^{(k+1)\,b}}{\partial x} + \frac{\partial u_x^{(k+1)\,b}}{\partial z}\right) + \widetilde{C}_{44}^{(k+1)}\left(\frac{\partial u_z^{(k+1)\,b}}{\partial y} + \frac{\partial u_y^{(k+1)\,b}}{\partial z}\right)\\ \sigma_{zy}^{k\,t} &= \widetilde{C}_{45}^{k}\left(\frac{\partial u_z^{k\,t}}{\partial x} + \frac{\partial u_x^{k\,t}}{\partial z}\right) + \widetilde{C}_{44}^{k}\left(\frac{\partial u_z^{k\,t}}{\partial y} + \frac{\partial u_y^{k\,t}}{\partial z}\right)\end{aligned} \tag{4}$$

In the most general case the fiber orientations and/or material properties of layer k are different than the ones relative to layer $k+1$. Thus, in the general case it can be inferred that

$$\widetilde{C}_{ij}^{(k+1)} \neq \widetilde{C}_{ij}^{k} \tag{5}$$

where $\widetilde{C}_{ij}^{(k+1)}$ is the generic Hooke's coefficient relative to layer $k+1$ whereas $\widetilde{C}_{ij}^{k}$ is the generic Hooke's coefficient relative to layer k. At any arbitrarily given point $P_1 \equiv (x_1, y_1)$ located at the interface between two consecutive layers the displacement u_z must be a continuous function in the thickness direction: $u_z^{(k+1)\,b}(x_1, y_1) = u_z^{k\,t}(x_1, y_1)$. At any other arbitrary point $P_2 \equiv (x_2, y_2)$ at the interface between the same two layers the displacement u_z must still be a continuous function: $u_z^{(k+1)\,b}(x_2, y_2) = u_z^{k\,t}(x_2, y_2)$. Since the points P_1 and P_2 can be selected anywhere in the x, y plate domain (note that the z coordinates of the two points correspond to $z_{\text{bot}_{(k+1)}}$ which is coincident with z_{top_k} by definition of interface) it is deduced that *all* the in-plane derivatives of the displacement u_z must also be continuous functions in the thickness direction (otherwise a satisfied continuity of the displacement u_z in P_1 would not imply the continuity of the same displacement in another point P_2 selected on the interface). Thus, the compatibility of the displacement u_z implies that *any derivative* of u_z of any order with respect to any direction *contained in the plane at the interface* between any two consecutive layers must be a continuous function along the thickness. A particular case of this statement is the continuity of the first derivatives (but as discussed above any order of in-plane derivatives must be continuous functions):

$$[u_z]_{z=z_{\text{bot}_{(k+1)}}} = [u_z]_{z=z_{\text{top}_k}} \Longleftrightarrow u_z^{(k+1)\,b} = u_z^{k\,t} \Rightarrow \begin{cases}\frac{\partial u_z^{(k+1)\,b}}{\partial x} = \frac{\partial u_z^{k\,t}}{\partial x}\\ \frac{\partial u_z^{(k+1)\,b}}{\partial y} = \frac{\partial u_z^{k\,t}}{\partial y}\end{cases} \tag{6}$$

where for example the following definition has been used:

$$\left[\frac{\partial u_z}{\partial x}\right]_{z=z_{\text{bot}_{(k+1)}}} \equiv \frac{\partial u_z^{(k+1)\,b}}{\partial x} \tag{7}$$