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# Influence of the edge-boundary conditions on three-dimensional free vibrations of isotropic and cross-ply multilayered rectangular plates

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ABSTRACT

This paper analyses the influence of different sets of edge-boundary conditions on the dynamics of freely vibrating isotropic and cross-ply multilayer laminated rectangular plates. The analysis is carried out within the frame of the full three-dimensional theory of elasticity through a formulation which is based on assumed displacements only; this formulation presents its relevant objectives in a unified manner, regardless of the nature of the stacking patterns of the laminated plates (isotropic, single layers or multi-layers). The analytical and/or numerical performance of the formulation is compared to those few results achievable through the exact three-dimensional theory and/or to those few existing results achieved by other researchers through alternative formulations. Convergence analyses are carried out on eigenvalues, displacement and stress fields in order to describe the capability of the formulation when compared to the exact three-dimensional results. The analysis reveals an interesting dependence on the edge-boundary conditions and highlights the need to carry out deeper investigations even though certain classical boundary conditions are taken into account through the most modern electronic computers.



## 1. Introduction

With respect to the several theories regarding the analysis of freely vibrating laminated plates spanning from the classical one to the higher order models (e.g. Refs. [1,2]), the three-dimensional theories provide an indispensible land mark. As far as three-dimensional studies are concerned, the scientific literature is classically indebted to the analysis provided by Srinivas et al. [3] from which, several researchers have taken advantage of the numerical results provided by these authors for an uncountable number of validations. Batra and Aimmanee [4] later on also complemented the work by Srinivas et al. [3].

The above mentioned analyses [3,4] regarding a particular class of edge-boundary conditions (i.e. simply supported plates) allow to achieve eigenproblem formulations from which exact eigenvalues could be extracted. This is done by opportunely adopting a suitable set of spatial circular functions.

Even though the simply supported boundary conditions allow a straightforward analytical treatment of the problem, they are not the only way to approximate the several real external conditions imposed at the edges of a plate; in truth a simply supported boundary condition is also afar from an easy realization in laboratory. Other classical boundary conditions, along with the simply supported condition, represent a slightly more enlarged feasible reality: clamped, simply supported, free and combinations. It is stressed that the realization of a "real" model (adherent to the respective "real" world) often needs an identification of the true boundary conditions which are usually in the middle of the mentioned classical ones (clamped: C, simply supported: S, free: F); however, in the present circumstances these boundary conditions are raised to a theoretical importance for comparison purposes. In this regard, the validation of a recent theory introduced by this author through Refs. [5,6] represents a part of the objectives of the present work due to the fact that the present work also aims (i) at showing how the accuracy of the dynamics of freely vibrating plates can be influenced by different boundary conditions and, moreover, (ii) at showing in a unified manner its relevant results for both isotropic and multilayer laminated plates; in particular, for this latter reason the present work should be intended to be a novelty with respect to the class of edge-boundary conditions herein investigated when compared with the existing scientific literature. Indeed, the literature lacks relevant studies for three-dimensional analyses involving boundary conditions which are different from simply supported along with a multilayered architecture. As a further objective (iii) the present study aims at analysing the convergence analysis through an added novelty with respect to the previous investigations; such a novelty makes certain convergence processes for a part of classical boundary conditions debatable and even though simple material arrangement (single isotropic layers) are accounted for. In order to appreciate the need to carry out the present investigation, the following chronological bibliographical discussion, within the

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frame of a three-dimensional analysis of rectangular plates, is worthy of mention.

Cheung and Chakrabarti [7] developed the finite layer method to investigate free vibrations of thick rectangular plates; the method, based on dividing the solid into a number of layers one on top of the other permitted the evaluation of frequencies quite close to the exact ones achieved by Srinivas et al. [3]. The existing numerical differences, even though in comparison to the existing exact case, were neither discussed nor investigated; the versatility of the method permitted the authors to evaluate natural frequencies for isotropic plates subjected to different boundary conditions and for a three-layered simply supported sandwich plate. Hutchinson and Zillmer [8] and Leissa and Zhang [9] dealt with three-dimensional solids (parallelepiped) which, within the frame of the three-dimensional phenomena, should deserve a certain mention. In particular, Hutchinson and Zillmer [8] investigated a completely free isotropic parallelepiped and were able to extract natural frequencies by adopting a series solution of the general equations of linear elasticity. They claim a relevant agreement between their results with certain previous experimental results. Leissa and Zhang [9] applied Ritz's method to investigate isotropic cantilevered rectangular parallelepipeds; this study was accompanied by a convergence analysis in line with the potentiality of the calculators of the age (in the average two significant digits were of comparable accuracy by changing the size of the problem); the study concluded in favour of possible straightforward extensions of the method for the calculation of stresses.

Liew et al. [10–14] carried out an extended investigation in the field. In particular, the works [10–12] were all based on a three-dimensional Ritz formulation whilst the works referred to Refs. [13,14] analysed the dynamics of freely vibrating plates through a numerical procedure based on differential quadrature. Apart from work [14], which considered homogeneous single layer orthotropic plates, all the works analysed isotropic homogeneous single layer plates whilst all the works [10–14] reported a convergence analysis based on frequencies. In particular, Ref. [10] noted a certain invariance of Ritz's formulation with respect to different boundary conditions. Any attempt aimed at verifying the fulfilment of related boundary conditions was not carried out.

Filipich et al. [15] extended a variational approach (named WEM by the authors) to the three-dimensional analysis of rectangular homogeneous isotropic plates and compare their results well with the exact ones for simply supported cases. This analysis was later on extended in Rosales et al. [16] where rectangular plates clamped at four lateral boundaries were analysed. In this latter reference Rosales et al. [16] showed eigenvalues (Table 1) slightly lower than those published by Liew et al. [10]. In any case, still homogeneous isotropic single layer plates were taken into account and an analysis aimed at verifying the fulfilment of any boundary condition was not carried out.

Zhou et al. [17], approximately 1 year later than Ref. [16] and 10 years later than Ref. [10], revisited the analysis of three-dimensional vibrations of isotropic thick rectangular plates by implementing Ritz's method through the Chebyshev polynomials multiplied by a boundary function. The accuracy of the analysis in this work is sensibly higher than that comparably achieved in Liew et al. [10]; this higher accuracy can be essentially attributed to the higher number of admissible polynomials used in Zhou et al. [17] and, therefore, in line with the improved calculators of the respective ages. In Ref. [17], homogeneous isotropic single layer plates were again taken into account along with a missing analysis aimed at verifying the fulfilment of any boundary condition.

Recently, Civalek [18] and Nagino et al. [19] analysed the mentioned three-dimensional problem. They used two different methods; in particular, Civalek [18] used a method based on the discretisation of both the governing differential equations of the motion and the related boundary conditions (the so called DSC-method, for Discrete Singular Convolution). The DSC-approach allowed the authors [18] to achieve results (frequencies) agreeing well with those obtained by other researchers; the comparisons also showed slight differences with regard to exact values for simply supported plates. The approach by Nagino et al. [19] consisted in Ritz's method along with B-spline functions; because the B-spline functions alone were not able to satisfy the essential boundary conditions, they used artificial springs in conjunction with B-spline functions. Such a method allowed a certain flexibility in treating different boundary conditions although the spring parameters must be tuned to allow the related convergence process. Both these latter studies [18,19] analysed homogeneous isotropic single layer plates and compared eigenvalues to test the convergence capability of the methods.

The work by Ye [20] is, finally, worthy of mention. In this work the author dealt with three-dimensional analyses of single or multilayered rectangular plates subjected to clamped boundaries. The approach was in line with that used by Fan and Ye [21]. In Ref. [21] such an approach was designed for simply supported boundary conditions along with the possibility to solve the equations of the linear elasticity and by keeping the size of the eigenproblem fixed regardless of the number of layers involved. Ye [20] was then able to extend the approach [21] by adding the possibility to introduce clamped boundaries; such edge-boundary conditions were obtained by suppressing the edge displacements at a number of planes parallel to the mid-plane plate. This was achieved through the Lagrange multipliers. Ye [20], therefore, was able to evaluate natural frequencies for isotropic and multilayered plates and thus providing an interesting land mark for fruitful comparisons.

**Table 1**
Frequencies, $\Omega = \omega L_x^2\sqrt{\rho h/D}$, of isotropic square plates ($\nu$ = 0.3, $L_x/h$ = 10; SSSS).

| $N_x \times N_y \times N_z$ | $I_{SS}$ | $II_{SA-AS}$ | $III_{SA-AS}$ | $IV_{AA}$ | $V_{SS}$ | $VI_{SS}$ |
|---|---|---|---|---|---|---|
| 3 × 3 × 3 | 19.150 | 46.063 | 64.383 | 70.978 | 87.624 | 91.052 |
| 4 × 4 × 4 | 19.090 | 45.622 | 64.3834905 | 70.113 | 85.517 | 91.05200545 |
| 5 × 5 × 5 | 19.090 | 45.619 | 64.383 | 70.104 | 85.488 | 91.052 |
| 6 × 6 × 6 | 19.09005294 | 45.619 | 64.383 | 70.104 | 85.488 | 91.052 |
| 7 × 7 × 7 | 19.090 | 45.61927417 | 64.383 | 70.10407502 | 85.48755209 | 91.052 |
| 8 × 8 × 8 | 19.090 | 45.619 | 64.383 | 70.104 | 85.488 | 91.052 |
| 9 × 9 × 9 | 19.090 | 45.619 | 64.383 | 70.104 | 85.488 | 91.052 |
| 10 × 10 × 10 | 19.090 | 45.619 | 64.383 | 70.104 | 85.488 | 91.052 |
| 11 × 11 × 11 | 19.090 | 45.619 | 64.383 | 70.104 | 85.488 | 91.052 |
| 12 × 12 × 12 | 19.090 | 45.619 | 64.383 | 70.104 | 85.488 | 91.052 |
| 13 × 13 × 13 | 19.090 | 45.619 | 64.383 | 70.104 | 85.488 | 91.052 |
| 14 × 14 × 14 | 19.090 | 45.619 | 64.383 | 70.104 | 85.488 | 91.052 |
| 15 × 15 × 15 | 19.09005294 | 45.61927417 | 64.3834905 | 70.10407502 | 85.48755209 | 91.05200545 |
| Exact[a] | 19.09005294 | 45.61927417 | 64.3834905 | 70.10407502 | 85.48755209 | 91.05200545 |

[a] Evaluated as described in Messina [26].