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Three-dimensional solution of axisymmetric bending of functionally graded circular plates

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ABSTRACT

Based on three-dimensional theory, this paper investigates the axisymmetric bending of transversely isotropic and functionally graded circular plates subject to arbitrarily transverse loads using the direct displacement method. The material properties can arbitrarily vary along the thickness of the plate. The transverse load is expanded in the Fourier–Bessel series and superposition principle is then used to obtain the total response based on the results of each item of the series. For one item of the series of the load, we assume the distributions of the displacements in the radial direction and therefore only the distributions of the displacements in thickness direction are required to find. If the material properties vary in an exponential law, the exact solutions can be obtained for elastic simple support and rigid slipping support, which are satisfied on the every point of the boundaries. Moreover, the analytical solutions are also presented for simply supported and clamped conditions, which are satisfied using Saint Venant principle. Simultaneously, through the layerwise method a semi-analytical solution is proposed for the case of arbitrary variation of the material properties. Finally the numerical examples are presented to verify the proposed method.

1. Introduction

The bending of circular plates is the classical problem in the theory of elasticity. Circular plates are also applied widely in many fields. It is therefore attracted much researchers' attention for a considerable time. Recently functionally graded materials (FGMs), whose properties continuously varied with respect to the coordinates, have been developed to overcome the problem associated with the interfaces in traditional composite materials due to the abrupt change of the materials properties. The theory of plates are usually extended to study the FGM plates since they are very successful in homogeneous plates. For instance, Reddy et al. [1] studied the axisymmetric bending of FGM circular and annular plates and presented the relation of the Mindlin plate theory between Kirchhoff plate theory. After that, Reddy [2] also gave a finite element method for FGM plates based on the cubic shear deformation plate theory. Ma and Wang [3] investigated the axisymmetric bending and buckling of FGM circular plates and also presented the relation of the plate theory of three order shear deformation between the Kirchhoff plate theory. Najafizadeh and Heydari [4,5] obtained the closed form solution of the buckling of FGM circular plates subject to uniform loads in radial direction using variational principle based on the plate theory of high order shear deformation. The meshless method was also employed to investigate the statical and dynamical behavior of FMG plates based on the Mindlin plate theory by Sladek et al. [6]. Using Mindlin–Reissner plate theory, Nosier and Fallah [7] obtained the analytical solution of the axisymmetric and asymmetric behavior of FGM circular plates with various clamped and simply supported boundary conditions under mechanical and thermal loadings. Saidi et al. [8] explored the axisymmetric bending and buckling of FGM circular plate using three order shear deformation plate theory.

Apart from these above mentioned methods based on the theory of plates, Zhong and Shang [9] firstly presented the three-dimensionally exact solution of FGM plates using the state space method. Kashtalyan [10] also obtained the three-dimensionally elastic solution of FGM plates through the general solution of inhomogeneous media. Ootao and Tanigawa [11] presented a three-dimensional solution for transient thermal stresses of an orthotropic functionally graded rectangular plate. Xu and Zhou [12] obtained the three-dimensional elasticity solution of functionally graded rectangular plates with variable thickness. However, these methods are only available for the rectangular plates with four simply supported edges and the exponential distribution of material. Recently, Li et al. [13] presented the solution of pure bending of FGM circular plates using stress function based on three-dimensional theory of elasticity. Li et al. [14,15] also obtained the elastic

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solution of the axisymmetric bending of circular and annular plates subject to the polynomial loads of even order. Li et al. [16] also extended the above method to analyze functionally graded magneto–electro-elastic circular plates subjected to uniform load. Yang et al. [17] also presented an analytical solution of the bending of annular plates under uniform loading. Nie and Zhong [18] analyzed the free vibration of functionally annular sectorial plates using a semi-analytical method. Kordkheili and Naghdabadi [19] and Bagri and Eslami [20] investigated thermoelastic problems of functionally graded annular/circular plates. To authors' knowledge, no analytical solution is found for the bending of FGM circular plates subject to arbitrarily axisymmetric transverse load based on three-dimensional theory of elasticity.

For this purpose, this work investigates the axisymmetric bending of transversely isotropic and functionally graded circular plates subject to arbitrarily transverse loads using the direct displacement method. The material properties can arbitrarily vary along the thickness of the plate. If the material properties vary in an exponential law, the exact solutions are obtained for two boundary conditions, which are elastic simple support and rigid slipping support. Moreover, the analytical solutions are also presented for simply supported and clamped conditions. Simultaneously, a layerwise model is introduced and a semi-analytical solution is proposed for the case of arbitrary variation of the material properties. Finally the numerical examples are presented to verify the proposed method.

2. Formulation

2.1. The governing equations of axisymmetry

Fig. 1 shows a transversely isotropic FGM circular plate of a radius a, thickness h. The isotropic plane is parallel to the middle plane of the plate. The origin of the coordinates is located on the center of the top surface of the plate. Since all the physical quantities are independent on θ in the cylindrical coordinates (r, θ, z), the equilibrium equations and the relations of stresses and displacements can be rewritten as

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{\partial\sigma_{rz}}{\partial z}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=0,\quad \frac{\partial\sigma_{rz}}{\partial r}+\frac{\sigma_{rz}}{r}+\frac{\partial\sigma_{zz}}{\partial z}=0 \tag{1}$$

and

$$\begin{aligned}\sigma_{rr}&=c_{11}\frac{\partial u_r}{\partial r}+c_{12}\frac{u_r}{r}+c_{13}\frac{\partial u_z}{\partial z},\quad \sigma_{zz}=c_{13}\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)+c_{33}\frac{\partial u_z}{\partial z}\\ \sigma_{\theta\theta}&=c_{12}\frac{\partial u_r}{\partial r}+c_{11}\frac{u_r}{r}+c_{13}\frac{\partial u_z}{\partial z},\quad \sigma_{rz}=c_{44}\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right)\end{aligned} \tag{2}$$

in which σ_{ij} and u_i are the stresses and displacements, respectively, c_{ij} denotes the elastic coefficients, which vary along the thickness of the plate only, i.e., $c_{ij}=c_{ij}(z)$.

For the non-dimensionalized formulations, the following dimensionless quantities are introduced

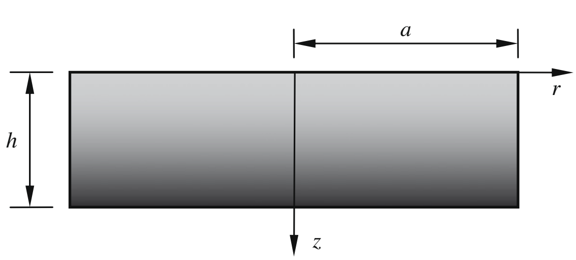

Fig. 1. FGM circular plate and coordinates system.

$$\begin{aligned}&\xi=r/a,\quad \zeta=z/h,\quad u=u_r/h,\quad w=u_z/h,\quad s=h/a\\ &\sigma_\xi=\sigma_{rr}/c,\quad \sigma_\theta=\sigma_{\theta\theta}/c,\quad \sigma_\zeta=\sigma_{zz}/c,\quad \tau_{\xi\zeta}=\sigma_{rz}/c,\quad \bar{c}_{ij}=c_{ij}/c\end{aligned} \tag{3}$$

where c is a constant with the dimension of stress, for instance, it can be taken $c=c_{11}(0)$. Then Eqs. (1) and (2) can be rewritten as

$$\frac{\partial\sigma_\xi}{\partial\xi}+\frac{1}{s}\frac{\partial\tau_{\xi\zeta}}{\partial\zeta}+\frac{\sigma_\xi-\sigma_\theta}{\xi}=0,\quad \frac{\partial\tau_{\xi\zeta}}{\partial\xi}+\frac{\tau_{\xi\zeta}}{\xi}+\frac{1}{s}\frac{\partial\sigma_\zeta}{\partial\zeta}=0 \tag{4}$$

and

$$\begin{aligned}\sigma_\xi&=s\left(\bar{c}_{11}\frac{\partial u}{\partial\xi}+\bar{c}_{12}\frac{u}{\xi}\right)+\bar{c}_{13}\frac{\partial w}{\partial\zeta},\quad \sigma_\zeta=s\bar{c}_{13}\left(\frac{\partial u}{\partial\xi}+\frac{u}{\xi}\right)+\bar{c}_{33}\frac{\partial w}{\partial\zeta}\\ \sigma_\theta&=s\left(\bar{c}_{12}\frac{\partial u}{\partial\xi}+\bar{c}_{11}\frac{u}{\xi}\right)+\bar{c}_{13}\frac{\partial w}{\partial\zeta},\quad \tau_{\xi\zeta}=\bar{c}_{44}\left(\frac{\partial u}{\partial\zeta}+s\frac{\partial w}{\partial\xi}\right)\end{aligned} \tag{5}$$

2.2. The expansion of the transverse loads

The transverse load acted on the top and bottom surfaces of the plate is denoted by $P(\xi)$. According to the Fourier–Bessel expansion law [21], if $k_1, k_2, k_3, \ldots$, denote the positive roots of $J_0(k)=0$ or $J_1(k)=0$ arranged in ascending order of magnitude, we have

$$P(\xi)=\sum_{i=1}^{\infty}C_iJ_0(k_i\xi) \tag{6}$$

where the coefficients C_i is

$$C_i=\begin{cases}\frac{2}{[J_1(k_i)]^2}\int_0^1P(\xi)J_0(k_i\xi)\xi d\xi & \text{if } J_0(k_i)=0\\ \frac{2}{[J_0(k_i)]^2}\int_0^1P(\xi)J_0(k_i\xi)\xi d\xi & \text{if } J_1(k_i)=0\end{cases} \tag{7}$$

where $J_\mu(k_i\xi)$ $(\mu=0,1)$ is the first kind Bessel function of order μ. Thus, if the solution of the load in the form $C_iJ_0(k_i\xi)$ is obtained, the bending of arbitrary transverse axisymmetric loads can be investigated through the superposition principle. As a result, we assume that the normal distribution load acted on the surfaces of the plate has the form $C_iJ_0(k_i\xi)$. In the following derivation, we use the symbol k to represent k_i for the sake of conciseness.

2.3. Direct displacement method

We assume that the two displacements are

$$u=J_1(k\xi)f(\zeta)+\xi u_1(\zeta),\quad w=J_0(k\xi)g(\zeta)+w_0(\zeta)+\xi^2w_2(\zeta) \tag{8}$$

where $f(\zeta), g(\zeta), u_1(\zeta), w_0(\zeta)$ and $w_2(\zeta)$ are the unknown and called displacement functions.

Substitution of Eq. (8) into Eq. (5) yields

$$\begin{aligned}\sigma_\xi&=(sk\bar{c}_{11}f+\bar{c}_{13}g')J_0(k\xi)+s(\bar{c}_{12}-\bar{c}_{11})f\xi^{-1}J_1(k\xi)\\ &\quad+s(\bar{c}_{11}+\bar{c}_{12})u_1+\bar{c}_{13}w_0'+\bar{c}_{13}\xi^2w_2'\\ \sigma_\theta&=(sk\bar{c}_{12}f+\bar{c}_{13}g')J_0(k\xi)+s(\bar{c}_{11}-\bar{c}_{12})f\xi^{-1}J_1(k\xi)\\ &\quad+s(\bar{c}_{11}+\bar{c}_{12})u_1+\bar{c}_{13}w_0'+\bar{c}_{13}\xi^2w_2'\\ \sigma_\zeta&=G(\zeta)J_0(k\xi)+2s\bar{c}_{13}u_1+\bar{c}_{33}w_0'+\xi^2\bar{c}_{33}w_2'\\ \tau_{\xi\zeta}&=F(\zeta)J_1(k\xi)+\xi\bar{c}_{44}(u_1'+2sw_2)\end{aligned} \tag{9}$$

where the prime denotes the derivative with respect to ζ and

$$G(\zeta)=\bar{c}_{33}g'+sk\bar{c}_{13}f,\quad F(\zeta)=\bar{c}_{44}(f'-skg) \tag{10}$$

Substituting Eq. (9) into Eq. (4) gives

$$\begin{aligned}&\{s^{-1}F'-k(sk\bar{c}_{11}f+\bar{c}_{13}g')\}J_1+\xi\{s^{-1}[\bar{c}_{44}(u_1'+2sw_2)]'\\ &\quad+2\bar{c}_{13}w_2'\}=0\end{aligned} \tag{11}$$