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Analytical beam theory for the in-plane deformation of a composite strip with in-plane curvature

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ABSTRACT

The variational-asymptotic-method (VAM) provides a mathematically rigorous way to reduce a three-dimensional elasticity formulation to a one-dimensional beam theory without *ad hoc* assumptions. In this work, the VAM is employed to develop a beam theory to analyze the in-plane deformation of a laminated strip-beam with initial in-plane curvature. The cross-sectional stiffness constants and recovery relations for stress and strain are presented as analytical expressions. For the case of zero initial curvature, consistency of the expressions with those of plate theory is demonstrated. For strip-beams with initial curvature in the in-plane direction, results obtained show explicit dependence on the curvature. Results are verified by comparison with those obtained from VABS, the accuracy and consistency of which with three-dimensional finite elements has been reported in several published works. In addition to the internal consistency check this work provides and its utility in helping to validate VABS (which is based on the principles of VAM), it is hoped that the results obtained herein, since they are all analytical expressions, will help researchers and engineers validate the effect of initial curvature in their beam theories, whether existing or new.

1. Introduction

The advent of composite materials has revolutionized the field of structural engineering, most notably due to their high strength-to-weight ratio and their directional tailorability. Modeling of structures with one dimension significantly larger than the other two as beams results in a much simpler mathematical formulation and helps save computational costs. With the advent of composites and structural members with initial twist/curvature, particularly in the field of aerospace engineering, using beam theories based on traditional approaches/ideas will not yield accurate results. Modeling of slender structural members with initial curvature is thus of paramount importance. The VAM provides a rigorous framework to model such structures without *ad hoc* assumptions regarding their deformation. The deformation is expressed in terms of an unknown set of warping functions, which is extracted using an asymptotic analysis of the variational problem using the system's inherent small parameters. The computer program VABS (Variational Asymptotic Beam Section) is constructed on the principles of the VAM.

One of first significant works concerning VABS was that of Ref. [1]. By the early 2000s, with the work of Ref. [2], VABS was established as an analysis tool of good standing in the circles of both academia and industry. Recent updates and developments to VABS are discussed in detail in [3,4]. Though novel ideas are not lacking in some of the beam theories in the current literature, the capability and generality of a VAM framework has maintained the superiority of VABS, subsequently making it a popular analysis tool for helicopter blades and wind turbines. Since then several efforts have contributed to the validation and verification of VABS results, and this paper is one such effort.

In this work, we propose a beam theory to analyze the in-plane deformation of an initially curved laminated strip-beam. A beam theory must address the following three aspects: a cross-sectional analysis leading to a stiffness matrix which is input into the 1D analysis, the 1D analysis itself, and the formulae or procedure to recover stress, strain and 3D displacement. This paper is organized as follows: Section 2 outlines the theoretical development leading to the results for the first and third aspects described previously. Section 3 demonstrates extraction of some stiffness terms using an equivalent plate theory. Section 4 validates the current work using results from VABS. Finally, conclusions are drawn.

2. Beam theory

Consider a laminated strip beam with initial curvature $k_3 = 1/R$ as shown in Fig. 1. This section deals with the development of a

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beam theory to describe the in-plane deformation of such a structural member. In the undeformed configuration, for a given axial coordinate (x_1), the unit vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ are defined to be tangent to the reference line and perpendicular to it as shown. Two frames of reference are used in the analysis to describe the deformed configuration. The 1D generalized strain measures associated with the frame of reference in which one of its unit vectors is tangent to the reference line are $\overline{\gamma}_{11}$ and $\overline{\kappa}_3$. On the other hand, those associated with the frame of reference in which one of its unit vectors is normal to the cross section are γ_{11}, κ_3 and $2\gamma_{12}$; geometrically exact expressions for both of these measures may be found in [5]. The kinematics development parallels that of [3], and the reader is advised to go through this reference for a complete description of the kinematics of this problem. Consequently, the strain expressions

$$\begin{aligned} \Gamma_{11} &= \frac{1}{\sqrt{g}}\left(\overline{\gamma}_{11} - x_2\overline{\kappa}_3 - k_3 w_2 + \frac{\partial w_1}{\partial x_1}\right) \\ \Gamma_{22} &= \frac{\partial w_2}{\partial x_2} \\ 2\Gamma_{12} &= \frac{1}{\sqrt{g}}\left(k_3 w_1 + \frac{\partial w_2}{\partial x_1}\right) + \frac{\partial w_1}{\partial x_2} \end{aligned} \tag{1}$$

are obtained, where the square root of the metric tensor of the undeformed state is given by

$$\sqrt{g} = 1 - x_2 k_3 \tag{2}$$

and where w_1 and w_2 are the unknown warping displacements. The problem we are dealing with is a plane stress problem; hence,

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \begin{Bmatrix} \Gamma_{11} \\ \Gamma_{22} \\ 2\Gamma_{12} \end{Bmatrix} \tag{3}$$

Consequently the strain energy per unit length is

$$\mathcal{U} = \frac{1}{2}\left\langle \sqrt{g} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}^T \begin{Bmatrix} \Gamma_{11} \\ \Gamma_{22} \\ 2\Gamma_{12} \end{Bmatrix} \right\rangle \tag{4}$$

where $\langle\ \rangle$ denotes an integration over the cross-section. Now define: $A_{ij} = \int_{-t/2}^{t/2} C_{ij}\,\mathrm{d}x_3$, i.e., an integration through the thickness, which can also be written as a summation over the various layers of the laminate (after appropriate coordinate transformations). One therefore can write the strain energy per unit length to be

$$\mathcal{U} = \int_{-c}^{c} \frac{1}{2}\sqrt{g} \begin{Bmatrix} \Gamma_{11} \\ \Gamma_{22} \\ 2\Gamma_{12} \end{Bmatrix}^T \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \Gamma_{11} \\ \Gamma_{22} \\ 2\Gamma_{12} \end{Bmatrix} \mathrm{d}x_2 \tag{5}$$

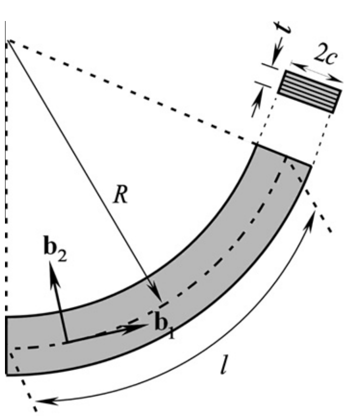

Fig. 1. Schematic of the composite strip beam with initial in-plane curvature.

This completes the formulation of the variational aspect of the problem. The current unknowns in the problem are the warping field. An attempt to solve this using standard variational principles will lead to the same difficulties as the corresponding elasticity problem. The solution of the problem is now carried out using asymptotic methods. One does this by identifying the inherent small parameters of the system: c/l and ck_3 which are assumed to be $O(\sigma)$. Also the maximum strain $(\max(\overline{\gamma}_{11}, c\overline{\kappa}_3) = O(\epsilon))$ is assumed to be small compared to unity. Before we proceed further, we define the following quantities which will be used in later analysis:

$$\begin{aligned} \overline{A}_{11} &= A_{11} + \frac{A_{22}A_{16}^2 - 2A_{12}A_{16}A_{26} + A_{12}^2 A_{66}}{A_{26}^2 - A_{22}A_{66}} \\ \widetilde{A}_{11} &= A_{22}A_{66} - A_{26}^2; \quad \widetilde{A}_{22} = A_{11}A_{66} - A_{16}^2; \quad \widetilde{A}_{66} = A_{11}A_{22} - A_{12}^2; \\ \widetilde{A}_{12} &= A_{16}A_{26} - A_{12}A_{66}; \quad \widetilde{A}_{16} = A_{12}A_{26} - A_{16}A_{22}; \\ &\quad \widetilde{A}_{26} = A_{12}A_{16} - A_{26}A_{11} \end{aligned} \tag{6}$$

The first step of the VAM is a zeroth-order or classical analysis where all terms $O(\sigma)$ are ignored in the strain energy. The warping is assumed to be of order $O(c\epsilon)$, and its subsequent solution justifies this assumption. Standard procedures of calculus of variations yield the warping as

$$\begin{aligned} w_1^{(0)} &= \frac{\widetilde{A}_{16}}{\widetilde{A}_{11}}\left(x_2\overline{\gamma}_{11} + \frac{c^2 - 3x_2^2}{6}\overline{\kappa}_3\right) \\ w_2^{(0)} &= \frac{\widetilde{A}_{12}}{\widetilde{A}_{11}}\left(x_2\overline{\gamma}_{11} + \frac{c^2 - 3x_2^2}{6}\overline{\kappa}_3\right) \end{aligned} \tag{7}$$

The classical strain energy per unit length is thus

$$\mathcal{U}_0 = \frac{1}{2}\begin{Bmatrix} \overline{\gamma}_{11} \\ \overline{\kappa}_3 \end{Bmatrix}^T \begin{bmatrix} 2c\overline{A}_{11} & 0 \\ 0 & \frac{2c^3}{3}\overline{A}_{11} \end{bmatrix} \begin{Bmatrix} \overline{\gamma}_{11} \\ \overline{\kappa}_3 \end{Bmatrix} \tag{8}$$

We then proceed to an analysis one order higher. This is done by perturbing the warping with terms of $O(\sigma c\epsilon)$. The resulting minimization problem leads to a set of Euler–Lagrange equations, which can be solved for the first-order warping

$$\begin{aligned} w_1^{(1)} &= \frac{\widetilde{A}_{16}\widetilde{A}_{12}}{6\widetilde{A}_{11}^2}k_3(c^2 - 3x_2^2)\overline{\gamma}_{11} + \frac{x_2 k_3}{6\widetilde{A}_{11}}\Big[\Big(2\overline{A}_{11}A_{26} - \widetilde{A}_{16} - \widetilde{A}_{26}\Big)c^2 \\ &+ \Big(\widetilde{A}_{26} - \widetilde{A}_{16}\Big)x_2^2\Big]\overline{\kappa}_3 \quad + \frac{A_{22}\overline{A}_{11} - \widetilde{A}_{66} + \widetilde{A}_{12}}{6\widetilde{A}_{11}}\overline{\gamma}'_{11} \\ &+ \frac{x_2}{6\widetilde{A}_{11}}\Big[\Big(-\overline{A}_{11}A_{22} - \widetilde{A}_{12} + \widetilde{A}_{66}\Big)(c^2 - x_2^2) + \overline{A}_{11}A_{22}(-3c^2 + x_2^2)\Big]\overline{\kappa}'_3 \\ w_2^{(1)} &= \frac{\widetilde{A}_{12}}{\widetilde{A}_{11}}\left(\frac{\widetilde{A}_{12}}{\widetilde{A}_{11}} - 1\right)k_3(c^2 - 3x_2^2)\overline{\gamma}_{11} \\ &\quad - \frac{x_2 k_3}{6\widetilde{A}_{11}}\Big[\Big(2\overline{A}_{11}A_{66} + \widetilde{A}_{22}\Big)c^2 + \Big(2\widetilde{A}_{12} - \widetilde{A}_{22}\Big)x_2^2\Big]\overline{\kappa}_3 \\ &\quad - \frac{\widetilde{A}_{12}\widetilde{A}_{16}}{6\widetilde{A}_{11}^2}(c^2 - 3x_2^2)\overline{\gamma}'_{11} \\ &\quad + \frac{x_2}{6\widetilde{A}_{11}^2}\Big[\widetilde{A}_{12}\widetilde{A}_{16}(c^2 - x_2^2) - \overline{A}_{11}\widetilde{A}_{11}A_{26}(x_2^2 - 3c^2)\Big]\overline{\kappa}'_3 \end{aligned} \tag{9}$$

Once the first-order warping is determined, the asymptotically correct second-order strain energy per unit length can be obtained