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Review

Recursive formulae for wave propagation analysis of FGM elastic plates via reverberation-ray matrix method

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ABSTRACT

Through a laminated plate model, a recursive formulation for the method of reverberation-ray matrix (MRRM) is proposed for the analysis of free wave propagation in an elastic plate with material properties varying along the thickness direction. In contrast to the traditional MRRM, the present new formulation behaves well for high frequency computation, while the heavy computational cost of storage and memory can also be cut down. Numerical examples are given to analyze dispersion characteristics of waves in FGM plates, and the comparison among traditional MRRM and the recursive formulation demonstrates the significant efficiency of the proposed recursive formula.

Contents

1. Introduction

Functionally graded materials (FGMs), such as particulate composites with continuously varying properties, were first designed for the purpose of reducing residual and thermal stresses and increasing the bonding strength in thermal-protection problems. Since then FGMs have gained widespread applicability in many fields such as aeronautics, astronautics, nuclear, biology, navigation, etc. [1–5]. Since good understanding of the material properties will make the use of FGMs more effective, the characteristic evaluation of the FGMs has long been a research subject. Aside from traditional laboratory testing methods based on the method of strength of materials, ultrasonic technique is another reliable method employed for material characterization. Therefore, more detailed study of wave propagation in FGMs is of considerable importance. However, it is generally very difficult to obtain exact analytical solutions for wave propagation in FGM structures except for a few special cases, since the material properties vary with the coordinates, adding obvious complexity to the governing equations. Hence, an appropriate multilayered model is usually applied to deal with the problems [6–10]. Actually the results will be accurate enough if there are adequate layers involved in the approximate laminate model.

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There have been extensive reports on wave propagation problems related to multilayered solids [11–16]. Various matrix formulations have been developed. The method of transfer matrix (MTM) formulated by Thomson [17] and slightly modified by Haskell [18] was soon widely applied for evaluating wave transmission in multilayered solid media. To overcome the inherent computational instabilities of MTM, a stiffness matrix method (SMM) using a layer stiffness matrix has been obtained for an isotropic medium by Kaussel and Roesset [19]. The spectral element method (SEM) originated from an analysis of wave propagation in bar and beam structures [20] could be an alternative approach.

Recently, a new matrix formulation method called the method of reverberation-ray matrix (MRRM), which was first developed by Pao et al. for studying transient waves in frames [21,22], has been successfully extended to analysis of wave propagation in layered isotropic solids [23], transversely isotropic laminas [24] and functionally graded elastic plates [25]. The calculation of dispersion curves in [25] demonstrates that MRRM possesses predominance in numerical stability compared to the conventional displacement method or the state-space method. However, with the increase of layer number in the approximate laminate model, the dimension of the reverberation-ray matrix in MRRM becomes larger, and hence, reducing its computational efficiency. The recent advances of MRRM can be found in the review papers by Pao et al. [26,27].

In this paper, we develop a recursive formulation (denoted as RF) of MRRM, which has the same dimension as the transfer matrix when the number of the layers increases. The recursive formulation is illustrated to investigate the dispersion behavior of waves in the FGM plates. Several numerical examples are performed, which demonstrate that the recursive formulation is unconditionally stable and more computationally efficient than the traditional MRRM.

2. Elastic waves in FGM plates

Consider an FGM plate with varying material properties in the thickness direction, as shown in Fig. 1. The total thickness is denoted by h. Since it is very difficult to obtain analytical solutions to the governing equations for wave propagation in an inhomogeneous plate, the plate can be equally divided into a multilayered plate, consisting of n layers, each with constant material properties. A global right-handed coordinate system (x,y,z) is adopted for the whole multilayered plate, whose origin is assumed to be located at the top surface. The individual layers are indicated by capital letters I, J, $K\ldots$, and the upper and lower surfaces of a typical layer, for instance the Jth layer with thickness h_J, are designated as the I and $J(I+1)$, respectively.

According to the elastodynamics theory [28], the differential equations governing the motion of an isotropic body with λ and μ being the Lamé constants are

$$(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})+\mu\nabla^2\mathbf{u}=\rho\partial^2\mathbf{u}/\partial t^2 \tag{1}$$

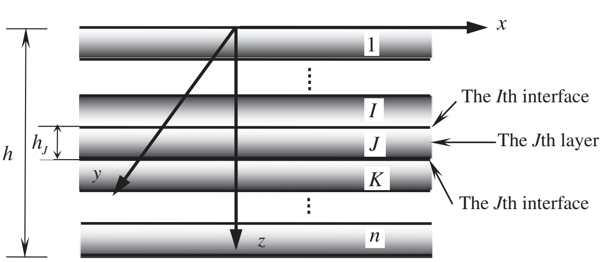

Fig. 1. The approximate laminate model of a FGM plate.

where $\mathbf{u}=[u,v,w]^T$ is the displacement vector, ρ is the material density, ∇ and ∇^2 are respectively the gradient operator and three-dimensional Laplacian. The constitutive relations are

$$\boldsymbol{\sigma}=\lambda(\nabla\cdot\mathbf{u})\mathbf{I}+\mu(\nabla\mathbf{u}+\mathbf{u}\nabla) \tag{2}$$

where $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{I}$ is the unit dyadic. According to Helmholtz theorem, the displacement field can be divided into an irrotational part and a rotational part as

$$\mathbf{u}=\nabla\phi+\nabla\times\boldsymbol{\psi},\quad \nabla\cdot\boldsymbol{\psi}=0 \tag{3}$$

where ϕ is the scalar potential, and $\boldsymbol{\psi}$ is the vectorial potential.

For plane strain problem, the displacement field $\mathbf{u}=[u,0,w]^T$ with the zero component in the y-direction, and the stress components $\boldsymbol{\sigma}$ are determined from the potentials ϕ and ψ as

$$u=\frac{\partial\phi}{\partial x}-\frac{\partial\psi}{\partial z},\quad w=\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial z} \tag{4}$$

and

$$\tau_{xz}=\mu\left(2\frac{\partial^2\phi}{\partial x\partial z}+\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial z^2}\right),$$
$$\sigma_{zz}=\lambda\nabla^2\phi+2\mu\left(\frac{\partial^2\phi}{\partial z^2}+\frac{\partial^2\psi}{\partial x\partial z}\right) \tag{5}$$

where

$$c_p^2\nabla^2\phi=\partial^2\phi/\partial t^2,\quad c_s^2\nabla^2\psi=\partial^2\psi/\partial t^2 \tag{6}$$

in which $\nabla^2=\partial^2/\partial x^2+\partial^2/\partial z^2$ is the two-dimensional Laplacian, $c_p=\sqrt{(\lambda+2\mu)/\rho}$ and $c_s=\sqrt{\mu/\rho}$ are respectively the velocities of P-wave and S-wave, and ψ is the non-zero component in $\boldsymbol{\psi}$, which has no dependence on y in this case.

Substituting the assumed wave solutions

$$\phi(x,z,t)=\hat{\phi}(z)e^{i(kx-\omega t)},\quad \psi(x,z,t)=\hat{\psi}(z)e^{i(kx-\omega t)} \tag{7}$$

into Eq. (6) gives

$$\frac{d^2\hat{\phi}(z)}{dz^2}+\alpha_p^2\hat{\phi}(z)=0,\quad \alpha_p=\sqrt{\omega^2/c_p^2-k^2},$$
$$\frac{d^2\hat{\psi}(z)}{dz^2}+\beta_s^2\hat{\psi}(z)=0,\quad \beta_s=\sqrt{\omega^2/c_s^2-k^2} \tag{8}$$

where k is the wave number in the x-direction, α_p and β_s are the waves numbers in the z-direction of P-wave and S-wave, respectively. Then the solutions to Eq. (8) can be expressed respectively as

$$\hat{\phi}(z)=a_1e^{i\alpha_pz}+d_1e^{-i\alpha_pz},\quad \hat{\psi}(z)=a_2e^{i\beta_sz}+d_2e^{-i\beta_sz} \tag{9}$$

where a_i and $d_i(i=1,2)$ are unspecified amplitudes of various waves. Thus the displacements and stresses expressions in Eqs. (4) and (5) can be rewritten as

$$u(x,z,t)=[ik(a_1e^{i\alpha_pz}+d_1e^{-i\alpha_pz})-i\beta_s(a_2e^{i\beta_sz}-d_2e^{-i\beta_sz})]e^{i(kx-\omega t)},$$
$$w(x,z,t)=[i\alpha_p(a_1e^{i\alpha_pz}-d_1e^{-i\alpha_pz})+ik(a_2e^{i\beta_sz}+d_2e^{-i\beta_sz})]e^{i(kx-\omega t)}, \tag{10}$$

$$\sigma_{zz}(x,z,t)=[-\mu(\beta_s^2-k^2)(a_1e^{i\alpha_pz}+d_1e^{-i\alpha_pz})-2\mu k\beta_s(a_2e^{i\beta_sz}-d_2e^{-i\beta_sz})]e^{i(kx-\omega t)},$$
$$\tau_{xz}(x,z,t)=[-2\mu k\alpha_p(a_1e^{i\alpha_pz}-d_1e^{-i\alpha_pz})+\mu(\beta_s^2-k^2)(a_2e^{i\beta_sz}+d_2e^{-i\beta_sz})]e^{i(kx-\omega t)} \tag{11}$$

3. A recursive formulation for MRRM

Consider two adjacent layers, say the J and Kth layers, for illustration. For each layer we select a set of dual local coordinates as depicted in Fig. 2. The x-directions of the two coordinates are the same, the y-directions are opposite to each other, and the z-directions follow the right-handed rule. The two superscripts of