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Three-dimensional elasticity solution of functionally graded rectangular plates with variable thickness

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ABSTRACT

This paper studies the stress and displacement distributions of continuously varying thickness functionally graded rectangular plates simply supported at four edges. Young's modulus is graded through the thickness following the exponential-law and Poisson's ratio keeps constant. On the basis of three-dimensional elasticity theory, the general expressions for the displacements and stresses of the plate under static loads, which exactly satisfy the governing differential equations and the simply supported boundary conditions at four edges of the plate, are analytically derived. The unknown coefficients in the general expressions of the stresses are approximately determined by using the double Fourier sinusoidal series expansions to the boundary conditions on the upper and lower surfaces of the plates. The effect of Young's modulus varying rules on the displacements and stresses of functionally graded rectangular plates is investigated. The proposed three-dimensional elasticity solution can be used to assess the validity of various approximate solutions and numerical methods for functionally graded plates.

1. Introduction

Functionally graded materials (FGMs) were first developed by a group of Japanese scientists to address the needs of aggressive environment of thermal shock [1]. Now, the concept of FGM has been widely explored in various engineering applications including electron, chemistry, optics, biomedicine and so on [2]. FGMs possess properties that vary continuously as a function of position within the material, thus FGMs can be used to avoid interfacial stress concentrations appeared in laminated structures. It should be mentioned that the classical thin plate theory [3–5] holds the Kirchhoff hypothesis that neglects shear deformation in the plate, which is increasingly significant when the plate becomes thicker. In comparison, the Mindlin plate theory [6] accounts for the shear deformations by introducing a shear correction factor, but it is limited to moderately thick plates. Although the higher-order plate theories [7] enhances the solution accuracy feasibly, only part of the elastic constants are considered, leading to the fact that the results will remain the same regardless of the variations of the elastic constants that are not included in the theory. The most common features of these simplified theories lies in that the effect of transverse normal stress is ignored, due to which the results are bounded to inherent errors for extremely thick plates [8]. To assess validity and establish accuracy of these and other approaches, which use a number of simplifying assumptions and hypotheses about stress and displacement fields in functionally graded plates, and also of numerical methods used in the analysis of such plates, exact analytical solutions based on the three-dimensional theory of elasticity are needed for some benchmark problems such as, e.g., bending of a rectangular plate. Zhong and Shang [9] developed a three-dimensional analysis for a rectangular plate made of orthotropic functionally graded piezoelectric material. The plate is simply supported and grounded along its four edges, and mechanical and electric properties of the material are assumed to have the same exponent-law dependence on the thickness coordinate. A three-dimensional solution of the coupling electroelastic fields in the plate under mechanical and electric loading on the upper and lower surfaces of the plate was obtained using state space approach. Kashtalyan [10] developed three-dimensional elasticity solution for a functionally graded simply supported plate subjected to transverse loading. Young's modulus of the plate is assumed to vary exponentially through the thickness, and Poisson's ratio is assumed to be constant. Huang et al. [11] presented the benchmark solutions for functionally graded thick plates resting on Winkler–Pasternak elastic foundations, the plate is assumed isotropic at any point in the plate volume, with Young's modulus varying exponentially through the thickness while Poisson's ratio remains the constant. Liew et al. [12] and Liew and Teo [13] presented the three-dimensional static and vibration solutions for thick rectangular plates by using the differential quadrature method. Moreover, Liew et al. [14] developed a continuum three-dimensionally

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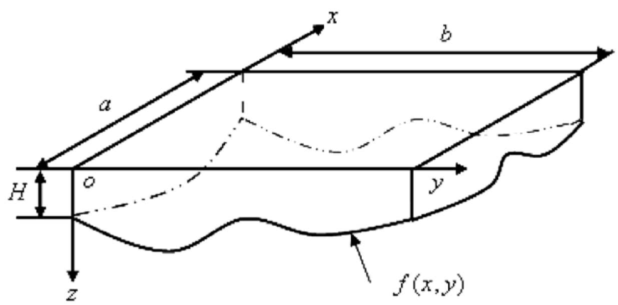

Fig. 1. Functionally graded rectangular plate with continuously varying thickness.

vibration analysis for thick rectangular plates based on the Ritz method. A literature survey on thick plate vibrations was given by Liew et al. [15].

Although plates with constant thickness have being widely used, the variable thickness plates have also received a lot of attention from designers and researches. The investigation on plates with varying thickness has significance in actual engineering because such plates can enhance the material potential through decreasing the self-weight and improving the distributions of stresses and displacements. For the elastic analysis of plates of variable thickness, only a limited number of closed-form solutions are known. Conway [16,17] studied the elastic bending of tapered axisymmetric plates. Ohga and Shigematsu [18] used a combination of boundary element and transfer matrix methods to solve variable thickness rectangular plates. This method provided a solution for only a special case of variable thickness rectangular plates. Fertis and his colleagues [19–22] developed a convenient and general method to solve variable thickness plates with various boundary conditions and loading by using equivalent flat plates. Zenkour [23] presented an exact solution for the bending of thin rectangular plates with uniform, linear, and quadratic thickness variations. Xu and Zhou [24] presented the three-dimensionally elasticity solution for simply supported rectangular plates with variable thickness. Based on the authors' knowledge, no three-dimensional elasticity solution to functionally graded rectangular plates with variable thickness has been reported.

In the present study, the general expressions for the displacements and stresses, which exactly satisfy the governing differential equations and simply supported conditions at four edges of the plates, have been analytically derived. The unknown coefficients in the stress expressions are approximately determined by the expansions of double Fourier sinusoidal series to the boundary conditions on the upper and lower surfaces of the plates. The proposed method has the generality and can be used to analyze the stress and displacement distributions of functionally graded rectangular plates with arbitrarily continuously varying thickness.

2. Elasticity solutions

Consider a continuously varying thickness functionally graded rectangular plate with length a, width b and thickness H at one side, as shown in Fig. 1. The plate is simply supported at four edges. The upper surface of the plate is horizontal and is subjected to the transverse load $q(x, y)$. The lower surface of the plate is described by the continuous functions $f(x, y)$. The plate is assumed isotropic at any point in the volume with constant Poisson's ratio μ, while Young's modulus E varies exponentially through thickness according to the following form

$$E = E_0^{e^{kz}}, \tag{1}$$

where k is the gradient index and E_0 is Young's modulus of the plate at $z = 0$. In the Cartesian coordinate system, the three-dimensional constitutive relations of an isotropic elastic body are given as follows:

$$\begin{aligned}
\sigma_x &= (\lambda+2G)\frac{\partial u}{\partial x}+\lambda\frac{\partial v}{\partial y}+\lambda\frac{\partial w}{\partial z}, \quad \tau_{yz} = G\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right),\\
\sigma_y &= (\lambda+2G)\frac{\partial v}{\partial y}+\lambda\frac{\partial u}{\partial x}+\lambda\frac{\partial w}{\partial z}, \quad \tau_{xz} = G\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right),\\
\sigma_z &= (\lambda+2G)\frac{\partial w}{\partial z}+\lambda\frac{\partial u}{\partial x}+\lambda\frac{\partial v}{\partial y}, \quad \tau_{xy} = G\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)
\end{aligned} \tag{2}$$

where σ_x, σ_y and σ_z are the normal stress components in the x, y and z directions. τ_{xy}, τ_{xz} and τ_{yz} are the shear stresses. u, v, w are the displacement components in the x, y and z directions respectively. λ, G are the Lamè constants with the same variation scheme of Young's modulus E.

Namely,

$$\lambda(z) = \lambda_0 e^{kz}, \quad G(z) = G_0 e^{kz}, \tag{3}$$

in which,

$$\lambda_0 = \frac{\mu E_0}{(1+\mu)(1-2\mu)}, \quad G_0 = \frac{E_0}{2(1+\mu)} \tag{4}$$

In the absence of body forces, the equilibrium equations of functionally graded rectangular plates can be written in terms of displacements as follows:

$$\begin{aligned}
&(\lambda_0+2G_0)\frac{\partial^2 w}{\partial z^2}+G_0\frac{\partial^2 w}{\partial x^2}+G_0\frac{\partial^2 w}{\partial y^2}+(\lambda_0+G_0)\frac{\partial^2 u}{\partial x\partial z}+(\lambda_0+G_0)\frac{\partial^2 v}{\partial y\partial z}\\
&\quad+k\lambda_0\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+k(\lambda_0+2G_0)\frac{\partial w}{\partial z}=0,\\
&(\lambda_0+2G_0)\frac{\partial^2 u}{\partial x^2}+G_0\frac{\partial^2 u}{\partial y^2}+G_0\frac{\partial^2 u}{\partial z^2}+(\lambda_0+G_0)\frac{\partial^2 v}{\partial x\partial y}+(\lambda_0+G_0)\frac{\partial^2 w}{\partial x\partial z}\\
&\quad+kG_0\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=0,\\
&(\lambda_0+2G_0)\frac{\partial^2 v}{\partial y^2}+G_0\frac{\partial^2 v}{\partial x^2}+G_0\frac{\partial^2 v}{\partial z^2}+(\lambda_0+G_0)\frac{\partial^2 u}{\partial x\partial y}+(\lambda_0+G_0)\frac{\partial^2 w}{\partial y\partial z}\\
&\quad+kG_0\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)=0.
\end{aligned} \tag{5}$$

For a rectangular plate simply supported at four edges, the boundary conditions are simulated by

$$\begin{aligned}
&\sigma_x = 0, \quad v = w = 0 \quad at x = 0, a,\\
&\sigma_y = 0, \quad u = w = 0 \quad at y = 0, b,
\end{aligned} \tag{6}$$

where a and b are the length and width of the plate, respectively.

Assume that the displacement distributions have the following form:

$$\begin{aligned}
u &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn}(z)\cos(\beta_m x)\sin(\gamma_n y),\\
v &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn}(z)\sin(\beta_m x)\cos(\gamma_n y),\\
w &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}(z)\sin(\beta_m x)\sin(\gamma_n y),
\end{aligned} \tag{7}$$

where $\beta_m = \frac{m\pi}{a}, \gamma_n = \frac{n\pi}{b}, U_{mn}(z), V_{mn}(z)$ and $W_{mn}(z)$ are the unknown functions about the coordinate z. It can be seen that Eq. (7) exactly satisfies Eq. (6).

Letting $\alpha_{mn} = \sqrt{\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}}$ and substituting Eq. (7) into Eq. (5), one has

$$\begin{aligned}
&G_0[U''_{mn}(z)+kU'_{mn}(z)-\alpha^2_{mn}U_{mn}(z)]-\beta^2_m(\lambda_0+G_0)U_{mn}(z)\\
&\quad-\beta_m\gamma_n(\lambda_0+G_0)V_{mn}(z)+\beta_m(\lambda_0+G_0)W'_{mn}(z)+\beta_m kG_0W_{mn}(z)=0,
\end{aligned} \tag{8}$$