Composite Structures 92 (2010) 1047–1051

Contents lists available at ScienceDirect

Composite Structures

journal homepage: www.elsevier.com/locate/compstruct

Free vibration of FGM plates with in-plane material inhomogeneity

D.Y. Liu [a], C.Y. Wang [b], W.Q. Chen [c,*]

[a] Department of Civil Engineering, Zhejiang University, Zijingang Campus, Hangzhou 310027, PR China
[b] Departments of Mathematics and Mechanical Engineering, Michigan State University, East Lansing, MI 48824, USA
[c] Department of Engineering Mechanics, Zhejiang University, Yuquan Campus, Hangzhou 310027, PR China

ARTICLE INFO

Article history:
Available online 7 October 2009

Keywords:
Functionally graded plate
In-plane inhomogeneity
Levy-type solution
Two-point boundary value problem
Initial value problem

ABSTRACT

An analysis is presented for the free vibration of a functionally graded isotropic elastic rectangular plate with in-plane material inhomogeneity. A Levy-type solution is obtained for plates with a pair of simply supported edges that are parallel with the material gradient direction. A particular integration method is adopted to solve the fourth-order ordinary differential equation with non-constant coefficients. The efficiency and accuracy of the analysis are demonstrated through numerical examples.

1. Introduction

Functionally graded materials (FGMs) are a new kind of materials exhibiting spatially continuous variation of material properties along one, two or three directions in a particular coordinate system. Since material interfaces are absent, the interfacial stress concentration phenomenon due to material mismatch as encountered in the conventional composite laminates or coated structures can be completely avoided. Primarily, FGMs were mainly developed as heat-resisting materials used in aerospace engineering. Recently, FGMs have also found wide applications in other areas, such as transducers, energy transform, biomedical engineering, optics, etc. [1].

There exists plenty of research on FGM structures (including beams, plates and shells) [2–17]. Most works deal with FGM structures with material inhomogeneity along the thickness direction only. When employing simplified structural theories (i.e. the beam, plate and shell theories), there is no significant difference in the analysis between FGM plates and the conventional laminated plates. Recently, Qian and Ching [18] used the meshfree local Petrov–Galerkin method to investigate the static and dynamic behavior of a cantilever beam with material properties varying along two directions. Goupee and Vel [19] performed an optimized natural frequency analysis of bi-directional FGM beams using the element-free Galerkin method. Lü et al. [20] presented a semi-analytical analysis of bi-directional FGM beams using the state-space based differential quadrature method [21–23]. In these works [18–20], the in-plane material inhomogeneity, in addition to the through-thickness one, has been taken into consideration, but only numerical or semi-analytical solutions have been obtained.

In this paper, the free vibration of a rectangular FGM plate with in-plane inhomogeneity is considered. The governing equation based on the classical plate theory is presented when the plate material is graded along one in-plane direction. For the plate simply supported at the edges parallel to that direction, a Levy-type solution is sought. The resulting ordinary differential equation is solved by a particular integration method which transforms a two-point boundary value problem to two initial value problems [24], which modified the method of Ref. [25]. Numerical results are finally given to indicate the accuracy and effectiveness of the present analysis.

2. Basic formulations

Consider a functionally graded isotropic elastic plate as shown in Fig. 1. We first assume that the plate has varying material properties as well as geometric properties in the plate plane $x-y$, i.e. the flexural rigidity of the plate $D = Eh^3/[12(1-\nu^2)] = D(x, y)$ is a function of x and y. The governing equation for free vibration can be deduced as

$$\nabla^2(D\nabla^2 w) - \left(\frac{\partial^2 D_\nu}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 D_\nu}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 D_\nu}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right) + \rho\frac{\partial^2 w}{\partial t^2} = 0 \tag{1}$$

where $D_\nu = D(1-\nu)$ and, at this stage, the Poisson's ratio ν and the density ρ can also be functions of x and y. If the Poisson's ratio ν is a constant, which is approximately true for metal–ceramic FGM, then the above equation reduces to

* Corresponding author. Tel.: +86 571 87951866; fax: +86 571 87952570.
E-mail address: chenwq@zju.edu.cn (W.Q. Chen).

0263-8223/$ - see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.compstruct.2009.10.001

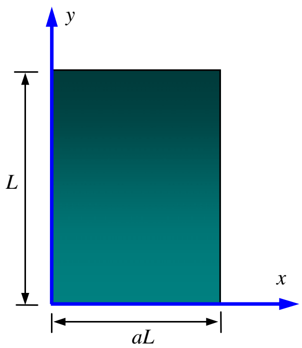

Fig. 1. A plate graded in the plate plane.

$$\nabla^2(D\nabla^2 w) - (1-\nu)\left(\frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 D}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right) + \rho\frac{\partial^2 w}{\partial t^2} = 0 \tag{2}$$

which is the same as that presented in the appendix of Ref. [26].

Now we confine ourselves to the particular case that the plate is graded in the y direction only, for which the material coefficients depend on y only. We also let the thickness of the plate be a constant h. Eq. (1) then becomes

$$D\nabla^4 w + \frac{\partial^2 D}{\partial y^2}\nabla^2 w + 2\frac{\partial D}{\partial y}\nabla^2\frac{\partial w}{\partial y} - \frac{\partial^2 D_\nu}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \rho\frac{\partial^2 w}{\partial t^2} = 0 \tag{3}$$

which is the governing equation of the FGM plate considered in this paper.

Assume that the plate material is made from ceramic and metal, and it is of full ceramic at $y = 0$ and of full metal at $y = L$. The material constants, such as the elastic modulus E, Poisson's ratio ν, and density ρ, satisfy the following relationship [27]

$$P = V_m P_m + V_c P_c, \quad V_m + V_c = 1, V_m = (y/L)^n \quad (0 \leqslant y \leqslant L) \tag{4}$$

where V_c and V_m are the volume fractions of ceramic and metal, and n, a real number, is the material graded index. This power law distribution of material property is one of the most appropriate and also simplest models for a two phase mixture, which is established by Voigt-type estimate [28].

If the gradient rule in Eq. (4) is employed, then

$$E = (y/L)^n E_m + [1-(y/L)^n]E_c \quad (0 \leqslant y \leqslant L) \tag{5a}$$
$$\nu = (y/L)^n \nu_m + [1-(y/L)^n]\nu_c \quad (0 \leqslant y \leqslant L) \tag{5b}$$
$$\rho = (y/L)^n \rho_m + [1-(y/L)^n]\rho_c \quad (0 \leqslant y \leqslant L) \tag{5c}$$

where subscripts m and c indicate metal and ceramic, respectively. Thus the expression for the flexural rigidity is

$$D = \frac{Eh^3}{12(1-\nu^2)} = \frac{(y/L)^n E_m + [1-(y/L)^n]E_c}{12\{1-[(y/L)^n \nu_m - [1-(y/L)^n]\nu_c]^2\}}h^3 \tag{6}$$

In general, the difference between the Poisson's ratios of metal and ceramic can be neglected, thus we have

$$D = \frac{(y/L)^n E_m + [1-(y/L)^n]E_c}{12(1-\nu^2)}h^3 \tag{7}$$

3. Solution

First we normalize all lengths by L, the length of the plate along the y-direction. The gradient rule in Eq. (5) becomes

$$E = E_0(1 - y^n + \mu y^n) = E_0\tau(y) \quad (0 \leqslant y \leqslant 1) \tag{8a}$$
$$\rho = \rho_0(1 - y^n + \beta y^n) = \rho_0\chi(y) \quad (0 \leqslant y \leqslant 1) \tag{8b}$$

where $\mu = E_m/E_c$, $\beta = \rho_m/\rho_c$, $E_0 = E_c$, and $\rho_0 = \rho_c$. Note that the same symbol y has been used for the dimensionless coordinate. Then the flexural rigidity in Eq. (7) becomes

$$D = \frac{Eh^3}{12(1-\nu^2)} = D_0\tau \tag{9}$$

where $D_0 = E_0h^3/[12(1-\nu^2)]$.Now we consider the case that the plate is simply supported at $x = 0$, a. The Levy-type solution can be sought by assuming

$$w = \sin(\alpha x)Y(y)e^{i\omega t} \tag{10}$$

where $\alpha = m\pi/a$.

Substituting Eq. (10) into Eq. (3), we derive

$$\tau Y^{(4)} + 2\tau' Y''' + (\tau'' - 2\alpha^2\tau)Y'' - 2\alpha^2\tau' Y' + (\alpha^4\tau - \nu\alpha^2\tau'' - k^4\chi)Y = 0 \tag{11}$$

where $k^4 = \rho_0\omega^2L^4h/D_0$.

The problem now reduces to solve Eq. (11), which is an ordinary differential equation with non-constant coefficients. If the solution is obtained, the lowest eigenvalue k, which is of most significance in engineering design, can be readily determined for given n, α, μ and β as well as the boundary conditions at $y = 0,1$. The solution however is not easy to find, and we shall employ the efficient method in Ref. [24], which turns the two-point boundary-value problem into two initial value problems.

If the plate is clamped at $y = 0$ and free at $y = 1$, the boundary conditions are

$$Y(0) = 0, \quad Y'(0) = 0 \tag{12a}$$
$$Y''(1) - \alpha^2\nu Y(1) = 0, \quad Y'''(1) - \alpha^2(2-\nu)Y'(1) = 0 \tag{12b}$$

where the first formula indicates zero deflection and zero rotation, and the second formula indicates zero moment and zero shear. Following Ref. [24], we assume that Y_1 and Y_2 satisfy Eq. (11) and the following initial conditions

$$Y_1(0) = 0, \quad Y_1'(0) = 0, \quad Y_1''(0) = 1, \quad Y_1'''(0) = 0 \tag{13a}$$
$$Y_2(0) = 0, \quad Y_2'(0) = 0, \quad Y_2''(0) = 0, \quad Y_2'''(0) = 1 \tag{13b}$$

A standard Runge–Kutta algorithm is adopted to obtain the solutions Y_1 and Y_2.

It is clear that the solution to Eq. (11) and (12a) can be expressed as a linear combination of the two independent solutions Y_1 and Y_2.

$$Y = C_1Y_1 + C_2Y_2 \tag{14}$$

By substituting Eq. (14) into Eq. (12b), we have two linear homogeneous algebraic equations.

$$\begin{pmatrix} Y_1''(1) - \alpha^2\nu Y_1(1) & Y_2''(1) - \alpha^2\nu Y_2(1) \\ Y_1'''(1) - \alpha^2(2-\nu)Y_1'(1) & Y_2'''(1) - \alpha^2(2-\nu)Y_2'(1) \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0 \tag{15}$$

For non-trivial solutions of C_1 and C_2, the determinant of coefficients of Eq. (15) should vanish. This leads to a nonlinear equation in the eigenvalue k, which can be obtained using numerical methods, such as the bisection technique. The lowest k gives the fundamental frequency of the plate as $\omega = (k/L)^2\sqrt{D_0/h\rho_0}$.

For the case where the edge $y = 0$ is clamped and the edge $y = 1$ is simply supported, Eq. (12b) is replaced by

$$Y(1) = 0, \quad Y''(1) = 0 \tag{16}$$

For the case where the edges $y = 0$, 1 are both clamped, Eq. (12b) is replaced by

$$Y(1) = 0, \quad Y'(1) = 0 \tag{17}$$