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An efficient and simple refined theory for bending and vibration of functionally graded plates

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ABSTRACT

A two-dimensional theory of functionally graded plates is presented using a mixed variational approach. The theory accounts for a displacements field in which the in-plane displacements vary linearly through the plate thickness, while the out-of-plane displacement is a second-degree function of thickness coordinate. The advantages of the present theory are that it contains both the transverse normal strain and stress in complete consistence with the boundary conditions at the top and bottom surfaces of the plates without loss of its simplicity. Therefore, the rationale for the shear correction factor used in such theories is obviated. The bending and free vibration problems of isotropic plates with material properties varying in the thickness direction are solved. Numerical results for frequencies are presented for two-phase graded material with a power-law through the plate thickness variation of the volume fractions of the constituents based on Mori–Tanaka scheme. In addition, numerical results of transverse deflections are obtained for FG simply supported isotropic plates with Young's modulus varying exponentially through the thickness and constant Poisson's ratio. The validity of the present theory is investigated by comparing some of the present results with their counterparts obtained due to three-dimensional approaches by Qian et al. and by Kashtalyan. The influence of the transverse normal strain on the bending and vibration of the FG plates is illustrated.

1. Introduction

Functionally graded materials (FGMs) are a class of composites that have a continuous variation of material properties from one surface to another. These materials can be fabricated by varying the percentage content of two or more materials such that the new materials have the desired property gradation in spatial directions. The gradation in the properties of the materials reduces thermal stresses, residual stresses and stress concentration factors found in laminated composites. FGMs have gained widespread applicability as thermal-barrier structures, wear- and corrosion-resistant coatings other than joining dissimilar materials. The concepts of FGMs were proposed by the Japanese Yamanouchi et al. [1] and Koizumi and Sata [2], and are projected as thermal barrier materials for applications in space planes, space structures and nuclear reactors.

Several studies have been performed to analyze the mechanical or the thermal or the thermo-mechanical responses of FG plates and shells. A comprehensive review is done by Tanigawa [3]. Reddy [4] has analyzed the static behavior of functionally graded rectangular plates based on his third-order shear deformation plate theory. Cheng and Batra [5] have related the deflections of a simply supported FG polygonal plate given by the first-order shear deformation theory and third-order shear deformation theory to that of an equivalent homogeneous Kirchhoff plate. Cheng and Batra [6] have also presented the results for the buckling and steady-state vibrations of a simply supported functionally graded polygonal plate based on Reddy's plate theory. Loy et al. [7] have studied the vibration of FG cylindrical shells using Love's shell theory. Qian et al. [8,9] employed the meshless local Petrov–Galerkin method to analyze free and forced vibrations of both homogeneous and FG thick plate based on the higher-order shear and normal deformable plate theory of Batra and Vidoli [10]. In this work, computed frequencies for a simply supported FG plate were found to match well with those obtained from the analytical solution of three-dimensional (3D) elasticity equations of Vel and Batra [11]. For a simply supported plate, Vel and Batra [11] used the classical plate theory, the first-order shear deformation (FSDT), and the third-order shear deformation (TSDT) approximations [12] for the displacement fields, assumed each displacement component to vary sinusoidally in the *x*- and the *y*-directions, and derived an algebraic equation for the frequencies. The assumed forms of displacements satisfy boundary conditions at the plate edges only when they are simply supported. Ferreira et al. [13] used the collocation multiquadric radial basis functions to analyze static deformations of simply

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supported FG plates modeled by a third-order shear deformation theory and a meshless method. Brischetto and Carrera [14] deduced advanced theories for bending analysis of FG plates using the Reissner mixed variational approach. Other recent studies on the two-dimensional models of FG plates may be found in [15–18].

Although there are several three-dimensional (3D) solutions available for inhomogeneous plates, most of these studies are for laminated plates consisting of homogeneous laminae [19,20]. Analytical 3D solutions for plates are useful since they provide benchmark results to assess the accuracy of various 2D plate theories and finite element formulations. Main and Spencer [21] established a class of exact 3D solutions for FG plates with traction-free surfaces. An asymptotic 3D theory of thermo-mechanical deformations of FG rectangular plates was developed by Reddy and Cheng [22]. Woo and Meguid [23] developed series solutions for large deflections of FG plates under transverse loading and a temperature field using von-Karman theory. Khoma [24] developed a method of constructing a general solution of the equilibrium equations for inhomogeneous transversely isotropic plates with elastic moduli that depend linearly on the transverse co-ordinate. Vel and Batra [25] presented a 3D solution for the cylindrical bending vibration of simply supported FG thick plates. They used displacement functions that identically satisfy boundary conditions to reduce the equations of motion to a set of coupled ordinary differential equations with variable coefficients, which are then solved by the power series method. Tsukamoto [26] examined thermal stresses in a ceramic–metal plate subjected to through-thickness heat flow using the Mori–Tanaka scheme and the classical laminated plate theory. Kashtalyan [27] obtained a 3D elasticity solution for a FG simply supported plates from the Plevako [28] general solution of the equilibrium equations for inhomogeneous isotropic media. Cheng and Batra [29] used an asymptotic expansion method to analyse three-dimensional thermoelastic deformations of FG elliptic plates, rigidly clamped at the edges. Power-law dependence of material properties on the thickness coordinate was assumed. The material properties were assumed to have power-law dependence on the thickness coordinate. Shen [30], and Yang and Shen [31] studied large deflections and postbuckling response of FG plates with temperature-dependent material properties using a classical plate theory and perturbation method.

A great deal of effort has been expended to keep the balance between an accurate representation and simplicity of formulation. Therefore, the equivalent single-layer theories for laminated or FG plates, from an engineering point of view, are still the most attractive approaches due to their simplicity and law computational cost.

The objective of this investigation is to present a refined equivalent single-layer shear deformation theory for orthotropic FG plates. This theory is obtained using a modified version of the mixed variational principle of Reissner [32,33]. The theory accounts for a displacements field in which the in-plane displacements vary linearly through the plate thickness, while the out-of-plane displacement is a parabolic function of thickness coordinate. The advantages of the present theory are that it contains both the transverse normal strain and stress in complete consistence with the boundary conditions at the top and bottom surfaces of the plates without loss of its simplicity. Therefore, the rationale for the shear correction factor used in such theories is obviated. The bending and free vibration problems of isotropic plates with material properties varying in the thickness direction are solved. Numerical results for frequencies are presented for two-phase graded material with a power-law through the plate thickness variation of the volume fractions of the constituents based on Mori–Tanaka scheme. In addition, numerical results of transverse deflections are obtained for FG simply supported isotropic plates with Young's modulus varying exponentially through the thickness and constant Poisson's ratio. The validity of the present theory is investigated by comparing some of its results with results obtained due to three-dimensional approaches by Qian et al. [34], and by Kashtalyan [27]. The influence of the transverse normal strain on the bending and free vibration responses of the FG plates is illustrated.

2. Theoretical formulation

An elastic plate of uniform thickness h, length a and width b is considered in a Cartesian coordinate system x, y and z. The plate is composed of an orthotropic elastic material which is smoothly gradient in the thickness direction. The plate occupies the following region:

$$0 \leqslant x \leqslant a, \quad 0 \leqslant y \leqslant b, \quad -h/2 \leqslant z \leqslant h/2.$$

Let the top surface $z = h/2$ and the bottom surface $z = -h/2$ of the plate be subjected to the following traction field:

$$\begin{aligned} \hat{t}(x,y,+h/2) &= (P_1^+, P_2^+, -q_1), \\ \hat{t}(x,y,-h/2) &= (P_1^-, P_2^-, +q_2). \end{aligned} \tag{1}$$

Thus, the boundary conditions on the plate surfaces are:

$$\begin{aligned} &\sigma_{33} = -q_1, \quad \sigma_{23} = P_2^+, \quad \sigma_{13} = P_1^+, \ \text{at } z = +h/2 \\ &\sigma_{33} = -q_2, \quad \sigma_{23} = P_2^-, \quad \sigma_{13} = P_1^-, \ \text{at } z = -h/2, \end{aligned} \tag{2}$$

where $\sigma_{ij} (i,j = 1,2,3)$ denote the stress components.

For convenience, the notation $q = q_2 - q_1$ will be used. Normal pressures are retained on both faces in order that one of them may, if desired, be made proportional to transverse displacement to simulate the effect of an elastic foundation.

For the present formulation, a modified version [32,33] of the mixed variational formula of Reissner is used for a composite elastic body with mixed boundary conditions such that a surface force F_i^* is prescribed over a part S_σ of the surface S of the body and a displacement u_i^* is prescribed over the remainder surface $S_u (S = S_\sigma + S_u)$; the mixed variational formula may be written in the form:

$$J = \int_{t_1}^{t_2} \left\{ \int\int_V\int \left(\frac{1}{2}\rho \dot{u}_i \dot{u}_i + R(\sigma_{ij}) - \sigma_{ij}\varepsilon_{ij} + F_i u_i \right) dV + \int_{S_\sigma}\int F_i^* u_i ds + \int_{S_u}\int n_j(\sigma_{ij} u_i) ds \right\} dt, \tag{3}$$

where ρ is the material density of the plate, F_i is the body force, n_j is a unit vector along the outward normal to the surface S, and $R(\sigma_{ij})$ is the complementary energy density which takes the form:

$$R(\sigma_{ij}) = a_{ijkl}\sigma_{ij}\sigma_{kl}, \tag{4}$$

where a_{ijkl} are the stiffness terms of the plate which depend on the material properties.

In the mixed variational formula (3), since the stresses and the displacements are taken to be arbitrary, the non-vanishing stresses components may be assumed without any dependence on the displacement components in the form:

$$\begin{aligned} \sigma_{ij} &= G_{ij}^{(1)}(x,y) + zG_{ij}^{(2)}(x,y), \quad (i,j = 1,2), \\ \sigma_{i3} &= G_{i3}(x,y)\left[1 - \left(\frac{z}{h/2}\right)^2\right] + \frac{P_i^+}{4}\left[c_1 + c_2\left(\frac{z}{h/2}\right)\right]\left[1 + \frac{z}{h/2}\right] \\ &\quad + \frac{P_i^-}{4}\left[c_3 + c_4\left(\frac{z}{h/2}\right)\right]\left[1 - \frac{z}{h/2}\right], \quad (i = 1,2), \\ \sigma_{33} &= \left(G_{33}^{(1)}(x,y) + zG_{33}^{(2)}(x,y)\right)\left[1 - \left(\frac{z}{h/2}\right)^2\right] + \sum_{n=1}^{4} z^{n-1} S_n(x,y). \end{aligned} \tag{5}$$