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# Untruncated infinite series superposition method for accurate flexural analysis of isotropic/orthotropic rectangular plates with arbitrary edge conditions

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**Abstract**

A new elegant, powerful and accurate superposition method is presented for isotropic/specially orthotropic rectangular plates with arbitrary transverse load and arbitrary combination of free/simply-supported/clamped/guided/elastically supported edges. All the component solutions used here are infinite series equivalents of the complicated closed-form Levy-type solutions employed in the conventional superposition method; it is shown that these series equivalents are easily derived. The mathematical equations pertaining to the various component solutions required for the application of this new method to any plate problem are clearly presented. A number of validation studies are carried out to verify the accuracy of the method. The method can be directly extended to the analysis of more complicated plates made of multifunctional or functionally graded materials.


*Keywords:* Rectangular plates; Orthotropic plates; Analytical solution; Superposition method

## 1. Introduction

Plate structures are encountered in all fields of engineering and hence there is a rich history of the literature on the analysis of plates of various shapes using a variety of methods [1–3]. Of all the available solutions, those based on an exact analytical approach, wherein the governing equations and the boundary conditions are satisfied rigorously, are valuable; there is renewed interest in such classical solutions originally developed for simple metallic plates because the solution methodologies are often applicable with minor changes to modern state-of-the-art laminated plate structures made up of functionally graded materials or those with magneto-electro-thermo-elastic coupling effects [4,5, for example]. Further, it is well-known that such computationally efficient analytical solutions are invaluable in identifying the influences of various structural and material parameters, and in subsequent optimization exercises. In this context, the objective of this work is to put forth an elegant alternative to the conventional superposition method, with specific reference to the title problem as a first step; this is done below after the relevant literature survey.

While simple closed-form solutions are available for some problems like symmetrical bending of isotropic circular plates or bending of a uniformly loaded clamped elliptical plate, the analysis of rectangular plates requires the use of convergent series. The simplest solutions are those of Navier for a plate simply supported on all edges and Levy for one with two opposite edges simply supported [3]. A wider class of boundary conditions involving clamped edges can be handled analytically by the classical superposition method wherein the above Navier or Levy solutions are appropriately superposed with solutions accounting for harmonically distributed moments applied on different edges; the method should be termed exact since a convergent Fourier series representation is valid for the fixed edge moments, and the boundary conditions can be satisfied to any degree of accuracy by taking an appropri-

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ate number of terms in the series. This method has been amply illustrated for static flexure problems by Timoshenko and Krieger [1] and for vibration and stability problems by Gorman [6] who also extended it to plates with free edges.

It has been shown [7] that both Navier and Levy methods can be simply extended to *specially orthotropic* plates (i.e. with the axes of orthotropy parallel to the edges) and symmetric cross-ply laminates; however, while no additional complexity is encountered in the former, the final form of the Levy solution for the orthotropic plate depends on the roots of a characteristic equation and may involve products of polynomial and hyperbolic functions or trigonometric and hyperbolic functions instead of just hyperbolic functions.

It is also known [8] that Navier-type solutions are possible for two unsymmetric laminate configurations – an unsymmetric cross-ply plate with simply supported edges of the shear diaphragm type (called S2 type), and an antisymmetric angle-ply plate with simply supported edges which permit the in-plane tangential displacement freely but completely restrain the in-plane normal displacement (called S3 type). The corresponding counterparts with two simply supported opposite edges are amenable to Levy-type analysis, but this becomes rather cumbersome because of the need to solve three coupled ordinary differential equations which lead to a high-degree characteristic equation with various possible combinations of real and distinct or repeated or complex roots; a formal solution using the state-space technique is often employed, but this would still involve Jordan canonical transformations depending on the nature of the roots [9].

While the simple Levy type solutions for isotropic homogeneous plates are easy to superpose for obtaining solutions for plates with general boundary conditions, the more complicated ones for homogenous or laminated anisotropic plates present serious difficulties for use with the classical superposition method. These difficulties have been discussed quite elaborately by Gorman and Ding [10,11] with reference to free vibration analysis; in fact, it has also been pointed out that additional difficulties arise in numerical calculations because of hyperbolic functions with large arguments. These numerical difficulties arise even for isotropic homogeneous plates as had been pointed out earlier [12]. As an easy way out of all the difficulties, a Galerkin type approximate procedure was employed in Refs. [10,11] for generating the component solutions for laminated plates.

The afore-mentioned difficulties, which mainly arise because of coupled multiple differential equations of high order and the associated possibility of various root combinations, would get aggravated when one employs a refined shear deformation theory because any such theory is of higher order as compared to the classical Kirchhoff–Poisson thin plate theory. This additional complexity is readily seen in the studies on symmetric cross-ply plates [13] and antisymmetric angle-ply plates [14] using the Mindlin-type first-order theory.

Very recently, it has been shown [15] that an exact infinite Fourier series counterpart of the closed-form Levy type solution can be generated easily for an isotropic/specially orthotropic plate subjected to a harmonically varying moment applied along any edge, and such component solutions are easy to work with in a superposition approach for the analysis of plates with any combination of clamped and simply supported edges. It has also been proved that no loss of accuracy occurs if the infinite series are summed without truncation, such untruncated summation being possible using mathematical packages like *MATHEMATICA* or *MATLAB*. Thus, the difficulties mentioned earlier with respect to the Levy-type solutions are completely avoided without resorting to any approximations.

The objective of the present paper is to show that the above approach, hereafter referred to as untruncated infinite series superposition method (UISSM), is quite powerful in that it can be employed successfully for the most general and most complicated plate problems involving any combination of simply supported, clamped, free, guided and elastically supported edges (denoted herein by S, C, F, G and E, respectively); further, the transverse load can be quite arbitrary – for example, it can be a partial load on the plate, or a line load/moment or a concentrated load applied on a free edge. All the required building blocks are derived here for the case of symmetric cross-ply plates within the purview of classical lamination theory; it is shown that Stokes' transformation [16] is necessary for generating some derivatives of the deflection function. Complete equivalence with the conventional superposition method involving closed-form Levy type solutions is proved. Finally, tabulated results are presented for some cases for future comparisons.

## 2. The new method – UISSM

Rectangular plates ($0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$) with an arbitrary combination of simply supported, clamped and free/guided edges are considered first; the plate may be isotropic homogeneous or a symmetric cross-ply laminate. The building blocks required in UISSM depend on the number of free/guided edges and hence are discussed separately under different categories as below. The extension of the method to account for elastic restraints is straightforward as explained later.

### *2.1. Category I: one free/guided edge (and also no free/guided edges)*

The building blocks for this category are shown in Fig. 1; they are valid when the bottom edge ($y = b$) is taken to be free. As can be seen, they correspond to a plate either simply supported all around or with one guided edge (i.e. one that is free to translate but with the normal slope completely restrained), and subjected to transverse load or edge moment or edge shear force expressed without loss of generality in the following form: