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Two-dimensional solutions for orthotropic materials by the state space method

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Abstract

A state space method for two-dimensional problems of orthotropic materials is proposed which provides an alternative method to investigate the mechanical behavior of homogeneous and laminated beams. The method is illustrated for a homogenous/laminated beam subjected to time-dependent transverse loads. A particular solution for a straight-crested wave propagating along an infinite beam is obtained. The bending, buckling and free vibrations of composite beams are also studied. Numerical results are presented and compared with those available in the literature.
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Keywords: State space method; Two-dimensional problem; Orthotropic material; Laminated beam

1. Introduction

The two-dimensional problem, which includes plane stress and strain problems, is very important in the field of elasticity. There are many practical problems of this kind in mechanical and civil engineering, such as a beam with rectangular section [1]. For thin isotropic and homogeneous beams, the classical beam theory and Timoshenko beam theory (the first-order shear theory) generally can give an excellent solution with negligible error [2].

Nowadays, much attention has been attracted by laminated composite beams because they have wide applications due to their distinct advantage over the conventional homogeneous beams. Because transverse shear deformation plays an important role in anisotropic or laminated beams, the classical beam theory based on Euler–Bernoulli hypothesis will lead to inaccurate or even erroneous results. Although Timoshenko beam theory takes into account the effects of transverse shear deformation and rotatory inertia, it is difficult to determine the required shear correction factors for arbitrary laminated beams. It has been shown that both the classical and first-order shear deformation theories are inadequate to predict the accurate solutions of laminated composite beams. Hence, various higher-order shear theories have been developed [3–6]. Karama et al. [7,8] presented an investigation of laminated composite beams by introducing an exponential function as a shear stress function. Some finite element formulations for laminated beams have also been established based on various hypotheses of displacements [9–13]. Recently, Chen et al. [14,15] proposed an interesting semi-analytical method by combining the differential quadrature method with the state space method and investigated the free vibration of cross-ply laminated beams and laminated plates in cylindrical bending [16]. They also successfully applied the method to generally laminated composite beams and plates [17–19].

This paper uses the state space method (also known as the method of initial functions) to study the two-dimensional problems of orthotropic materials. It should be noted that Das and Setlur [20] investigated two-dimensional elastodynamic problems of isotropic materials and obtained some interesting results such as an exact governing equation of the transverse vibration of a deep beam

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and a layered beam in plane stress by employing a similar method. However, as will be shown in this paper, the generalization of their method to orthotropic materials is not straightforward. The governing equation of a homogenous or laminated beam subjected to time-dependent transverse loads is derived. A particular solution of wave propagation in an infinite beam is obtained. The bending, buckling and free vibrations of homogeneous and composite beams are also studied. This investigation should provide an alternative method to derive approximate beam theories or directly give the exact solutions.

2. Mathematical formulations

In absence of body forces, the two-dimensional dynamic equilibrium equations of an elastic body with an initial strain ε_x^0 due to the initial stress σ_x^0 in x-direction are

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} = \rho \dfrac{\partial^2 u}{\partial t^2} + \sigma_x^0 \dfrac{\partial^2 u}{\partial x^2} \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} = \rho \dfrac{\partial^2 v}{\partial t^2} + \sigma_x^0 \dfrac{\partial^2 v}{\partial x^2} \end{cases} \tag{1}$$

where u and v are displacements in x- and y-directions, respectively, σ_x, σ_y and τ_{xy} are stresses, and ρ and t denote density and time, respectively. For the plane stress problem of orthotropic materials, the stress–displacement relations are

$$\begin{cases} \dfrac{\partial u}{\partial x} = s_{11}\sigma_x + s_{12}\sigma_y \\ \dfrac{\partial v}{\partial y} = s_{12}\sigma_x + s_{22}\sigma_y \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = s_{66}\tau_{xy} \end{cases} \tag{2}$$

where s_{11}, s_{12}, s_{22} and s_{66} are elastic compliances. Introduce the following notations

$$\begin{aligned} &U = u/H, \quad V = v/H, \quad Y = S\sigma_y, \quad X = S\tau_{xy} \\ &\{\mathbf{a}\} = [U \quad Y \quad V \quad X]^{\mathrm{T}}, \quad \beta = \partial/\partial\zeta, \quad \alpha = R\partial/\partial\eta, \\ &\xi^2 = \rho S H^2 \partial^2/\partial t^2 + m_2 \varepsilon_x^0 H^2 \partial^2/\partial x^2, \\ &\zeta = y/H, \quad \eta = x/L, \quad R = H/L \end{aligned} \tag{3}$$

and

$$\begin{aligned} &m_1 = s_{12}/s_{11}, \quad m_2 = S/s_{11}, \\ &m_3 = (s_{11}s_{22} - s_{12}^2)/(s_{11}S), \quad m_4 = s_{66}/S \end{aligned} \tag{4}$$

where L and H denote two length parameters which are prominent dimensions of the problem, and S is another parameter which has the same dimension as material compliance. Eqs. (1) and (2) can be rewritten as

$$\begin{cases} \beta\{\mathbf{a}\} = [\mathbf{M}]\{\mathbf{a}\} \\ \sigma_x = m_2\alpha U - m_1 Y \end{cases} \tag{5}$$

where

$$[\mathbf{M}] = \begin{bmatrix} 0 & \mathbf{A} \\ \mathbf{B} & 0 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -\alpha & m_4 \\ \xi^2 & -\alpha \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} m_1\alpha & m_3 \\ \xi^2 - m_2\alpha^2 & m_1\alpha \end{bmatrix} \tag{6}$$

Note that the differential operators α, β and ξ follow the usual rules of algebra [20].

The solution of the first one of Eq. (5) is

$$\{\mathbf{a}(\zeta)\} = \mathrm{e}^{[\mathbf{M}]\zeta}\{\mathbf{a}(0)\} \tag{7}$$

where $\mathbf{a}(0)$ denotes $\mathbf{a}(\zeta)$ at $\zeta = 0$, the initial unknowns on the plane $\zeta = 0$. Using the Cayley–Hamilton theorem, one has

$$\mathrm{e}^{[\mathbf{M}]\zeta} = \alpha_0[\mathbf{I}_4] + \alpha_1[\mathbf{M}] + \alpha_2[\mathbf{M}]^2 + \alpha_3[\mathbf{M}]^3 \tag{8}$$

where $[\mathbf{I}_n]$ is an identity matrix of $n \times n$ and the parameters α_i $(i = 0, 1, 2, 3)$ are determined by

$$\begin{cases} \mathrm{e}^{\gamma_1\zeta} = \alpha_0 + \alpha_1\gamma_1 + \alpha_2\gamma_1^2 + \alpha_3\gamma_1^3 \\ \mathrm{e}^{\gamma_2\zeta} = \alpha_0 + \alpha_1\gamma_2 + \alpha_2\gamma_2^2 + \alpha_3\gamma_2^3 \\ \mathrm{e}^{\gamma_3\zeta} = \alpha_0 + \alpha_1\gamma_3 + \alpha_2\gamma_3^2 + \alpha_3\gamma_3^3 \\ \mathrm{e}^{\gamma_4\zeta} = \alpha_0 + \alpha_1\gamma_4 + \alpha_2\gamma_4^2 + \alpha_3\gamma_4^3 \end{cases} \tag{9}$$

where γ_i $(i = 1, 2, 3, 4)$ are the distinct eigenvalues of the operator matrix [**M**]. If the material is isotropic, these eigenvalues can be expressed by the operators α and ξ explicitly as well as α_i $(i = 0, 1, 2, 3)$ [20]. Thus, $\mathrm{e}^{[M]\zeta}$ is a matrix whose elements also can be expressed by the operators α and ξ explicitly. Hence, Eq. (7) is an exact governing differential equation set of the plane problem.

Unfortunately, the eigenvalues of the operator matrix [**M**] cannot be obtained explicitly for anisotropic materials. However, by taking the advantage of the characteristic of the matrix [**M**] and the relations between the roots and coefficients of a quadratic equation with one unknown, all the elements of $\mathrm{e}^{[\mathbf{M}]\zeta}$ can be rendered by the differential operators α and ξ for orthotropic materials, as will be shown below.

Noticing Eq. (6), one has

$$[\mathbf{M}]^2 = \begin{bmatrix} 0 & \mathbf{A} \\ \mathbf{B} & 0 \end{bmatrix}\begin{bmatrix} 0 & \mathbf{A} \\ \mathbf{B} & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{C} & 0 \\ 0 & \mathbf{D} \end{bmatrix} \tag{10}$$

where

$$[\mathbf{C}] = [\mathbf{A}][\mathbf{B}] = \begin{bmatrix} m_4\xi^2 - (m_1 + m_2m_4)\alpha^2 & (m_1m_4 - m_3)\alpha \\ (m_1 - 1)\xi^2\alpha + m_2\alpha^3 & m_3\xi^2 - m_1\alpha^2 \end{bmatrix} \tag{11}$$

$$[\mathbf{D}] = [\mathbf{B}][\mathbf{A}] = \begin{bmatrix} m_3\xi^2 - m_1\alpha^2 & (m_1m_4 - m_3)\alpha \\ (m_1 - 1)\xi^2\alpha + m_2\alpha^3 & m_4\xi^2 - (m_1 + m_2m_4)\alpha^2 \end{bmatrix} \tag{12}$$

Obviously, all elements of matrix [**C**] are the same as the ones of matrix [**D**] except from the positions of the diagonal elements. Hence one has

$$\det([\mathbf{C}] - \lambda[\mathbf{I}_2]) = \det([\mathbf{D}] - \lambda[\mathbf{I}_2]) \tag{13}$$

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