Computers and Geotechnics 76 (2016) 1–11

Contents lists available at ScienceDirect

Computers and Geotechnics

journal homepage: www.elsevier.com/locate/compgeo

Research Paper

A weak form quadrature element formulation for coupled analysis of unsaturated soils

Shuai Yuan, Hongzhi Zhong *

Department of Civil Engineering, Tsinghua University, Beijing 100084, PR China

A R T I C L E   I N F O

Article history:
Received 8 August 2015
Received in revised form 30 December 2015
Accepted 14 February 2016
Available online 1 March 2016

Keywords:
Weak form quadrature element method
Unsaturated soil
Consolidation
Soil water characteristic curve

A B S T R A C T

A weak form quadrature element formulation is proposed for coupled analysis of unsaturated soils. In the formulation, integration points are coincident with nodes where all variables are obtained so that numerical implementation is made easy. Accuracy can be improved by the increase of the approximation order. An elasto-plastic constitutive model is used for description of the mechanical behavior, and hydraulic hysteresis of the soil water characteristic curve is considered. Results are compared with available analytical solutions, experimental data and numerical solutions in the literature, and good agreement is reached. Rapid convergence is reached as compared with the conventional finite element method. The effect of the ground water level is discussed.

1. Introduction

Despite the remarkable experimental and theoretical achievements that have been made for saturated soils [1], analysis of unsaturated soils has been still relatively less mature. Due to the ubiquity of unsaturated soils in geotechnical engineering, the development of relevant theories and simulation techniques for unsaturated soils becomes imperative.

One of the essential issues in analysis of unsaturated soils is the definition of the basic stress variables for the three-phased material. The concept of effective stress, which had been successfully introduced in analysis of saturated soils, was once the first choice. If the effective stress could be found and used solely for description of unsaturated soils, the established framework for saturated soils could have been easily extended to unsaturated soils. Although some forms of effective stress such as the Bishop stress [2,3] were explored and some success was made, the single effective stress was deemed inadequate for complete representation of unsaturated soils [4,5]. It has now been generally recognized that, at least two stress variables are needed and their form in different models remains a matter of convenience [6]. Among those available choices, the net stress and the suction are the most appropriate set of independent stress variables [7]. With the independent stress variables, stress paths and loading situations are easily described. However, saturated and unsaturated soils are hard to be depicted consistently because the effective stress cannot be obtained when soils become saturated. The other type of models uses a modified stress vector and an additional variable to represent the two different effects of suction, namely the change of the average skeleton stress and the suction-induced hardening [8]. Using some of those stress variables such as the Bishop stress and the suction, soils can be consistently modeled in the whole range of saturation and the hydro-mechanical coupling is directly introduced. Both groups of stress variables have been applied in the recent constitutive models. In fact, a constitutive model can be transformed and expressed in terms of any kind of stress variables as indicated by Sheng et al. [9,10].

Unsaturated soils were grouped into three categories by Barden [11]: highly saturated soils in which the air phase is in the form of bubbles and the fluid phase is continuous, lowly saturated soils in which the air phase is continuous and the fluid phase is discontinuous, and other soils where both the air and the fluid phases are continuous. The consolidation of highly saturated soils can be modeled based on the modified Biot's theory of saturated soils in which the fluid phase and the dissolved air bubbles are treated as a single material and the effect of air bubbles is considered through compressibility of the mixture. Some representative work in this area can be found in [12,13]. For soils with continuous air and fluid phases, Hasan and Fredlund [14], Lloret and Alonso [15] put forward the one-dimensional consolidation theory. Analytical solutions were given for single-layered and multi-layered unsaturated soils based on Fredlund's theory by Qin et al. [16] and Shan et al. [17,18], respectively. Darkshanamurthy et al. [19] proposed the three-dimensional theory for unsaturated consolidation with the net stress and the suction as the basic stress variables based

* Corresponding author. Tel.: +86 10 62781891.
E-mail address: hzz@tsinghua.edu.cn (H. Zhong).

http://dx.doi.org/10.1016/j.compgeo.2016.02.006
0266-352X/

on the linear constitutive model by Fredlund and Morgenstern [20]. Since then elasto-plastic analysis of unsaturated soils came into being [9,21–23].

The three phases in unsaturated soils imply that the degrees of freedom are increased greatly in analysis of unsaturated soils as compared with that of the single-phased material. Both mechanical and hydraulic hystereses exist in modeling of unsaturated soils. For these reasons, large computational resources are needed for numerical simulation of unsaturated soils. The weak form quadrature element method (QEM) is a high-order numerical algorithm which enjoys rapid convergence as compared with the conventional finite element method. In a finite element formulation, numerical integration is performed after element analysis during which strains and stresses of the element are estimated based on the selected shape functions for the element. In a weak form quadrature element formulation, however, the sequence of differentiation and numerical integration is reversed and element description (construction of shape functions) is therefore rendered unnecessary. A large number of degrees of freedom can be saved and the computational resources could be reduced significantly. In addition, less upfront data are required and the implementation of the QEM is rather straightforward. However, the semi-bandwidth of the coefficient matrix of a quadrature element formulation is large as compared with that of the conventional finite element method, which to some degree offsets the gain in problem size reduction. The QEM has been successfully applied in analyses of various structures [24–26] and in seepage and consolidation analyses [27,28] of saturated soils. In this paper, the QEM is employed for simulation of unsaturated soils under the assumption of small deformation. The effective stress and the suction are chosen as the stress variables. An effective stress-based elasto-plastic constitutive model is used for description of the mechanical behavior, and hydraulic hysteresis of the soil water characteristic curve is considered. Results are compared with available analytical solutions, experimental data and numerical solutions in the literature. Good agreement is reached, highlighting the computational effectiveness of the present formulation. The influence of the ground water level is also examined.

2. Formulation

2.1. Weak form description of unsaturated consolidation

In the following sections, boldfaced letters indicate matrices and vectors and boldfaced letters with subscripts i, j, k stand for their values at node (i, j, k). The pore fluid pressure u_w, pore air pressure u_a, the stress $\boldsymbol{\sigma}$ and the strain $\boldsymbol{\varepsilon}$ are all positive for compression. The differential equations for consolidation of unsaturated soils are composed of equilibrium of the soil and continuity of the pore fluid and air [29]:

$$\mathrm{div}\dot{\boldsymbol{\sigma}} - \dot{\mathbf{f}} = \mathbf{0} \tag{1}$$

$$\mathrm{div}\mathbf{v}_w^r = \psi\dot{\varepsilon}_v - \bar{a}_{11}\dot{u}_w + a_{12}\dot{u}_a \tag{2}$$

$$\mathrm{div}\mathbf{v}_a^r = (1-\psi)\dot{\varepsilon}_v - \bar{a}_{22}\dot{u}_a + a_{21}\dot{u}_w \tag{3}$$

with

$$\begin{aligned}&\bar{a}_{11} = c_w n_w + a_{12}, \quad \bar{a}_{22} = c_a n_a + a_{21}, \quad a_{12} = a_{21} = -n\frac{\partial S_r}{\partial s},\\ &\psi = S_r - n\frac{\partial S_r}{\partial \varepsilon_v}\end{aligned} \tag{4}$$

where the superposed dot denotes the material time derivative defined with respect to the soil skeleton, $\mathbf{f}$ is the body force which varies with time for variable saturation of the soil element or variable density of the pore fluid and air, $\mathbf{v}_w^r$ is the Darcian velocity of the pore fluid, n_w is the volumetric content of the fluid phase, ε_v is the volumetric strain, $\mathbf{v}_a^r$ is the Darcian velocity of the pore air, n_a is the volumetric content of the air phase, c_w and c_a are the respective compressibilities of the fluid and air phases, n is the porosity, S_r is the saturation with fluid and $s = u_a - u_w$ is the suction. The effective stress $\boldsymbol{\sigma}'$ is defined as

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - u_a\mathbf{M} + S_r(u_a - u_w)\mathbf{M} \tag{5}$$

where $\mathbf{M} = [1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0]^T$ is an auxiliary vector. The rate form of Eq. (5) is

$$\begin{aligned}\dot{\boldsymbol{\sigma}}' &= \dot{\boldsymbol{\sigma}} - \dot{u}_a\mathbf{M} + \left(S_r + \left(\frac{\partial S_r}{\partial s}\right)(u_a - u_w)\right)(\dot{u}_a - \dot{u}_w)\mathbf{M} + \frac{\partial S_r}{\partial \varepsilon_v}(u_a - u_w)\dot{\varepsilon}_v\mathbf{M}\\ &= \dot{\boldsymbol{\sigma}} - (A_1\dot{u}_a + A_2\dot{u}_w - A_3\dot{\varepsilon}_v)\mathbf{M},\\ A_1 &= \left(1 - S_r - \left(\frac{\partial S_r}{\partial s}\right)(u_a - u_w)\right),\\ A_2 &= \left(S_r + \left(\frac{\partial S_r}{\partial s}\right)(u_a - u_w)\right),\\ A_3 &= \frac{\partial S_r}{\partial \varepsilon_v}(u_a - u_w).\end{aligned} \tag{6}$$

It is noted that Eq. (6) is different from that in [23] where the rate of saturation was neglected. After introduction of Eq. (6) into Eq. (1), the condition of equilibrium becomes

$$\mathrm{div}(\dot{\boldsymbol{\sigma}}' + (A_1\dot{u}_a + A_2\dot{u}_w - A_3\dot{\varepsilon}_v)\mathbf{M}) - \dot{\mathbf{f}} = \mathbf{0} \tag{7}$$

The corresponding weak form is

$$\begin{aligned}&\int_V \delta\boldsymbol{\varepsilon}^T\dot{\boldsymbol{\sigma}}\mathrm{d}V - \int_V \delta\mathbf{u}^T\dot{\mathbf{f}}\mathrm{d}V - \int_S \delta\mathbf{u}^T\dot{\mathbf{q}}\mathrm{d}S = 0\\ &\Rightarrow \int_V \delta\boldsymbol{\varepsilon}^T(\dot{\boldsymbol{\sigma}}' + A_1\dot{u}_a\mathbf{M} + A_2\dot{u}_w\mathbf{M} - A_3\dot{\varepsilon}_v\mathbf{M})\mathrm{d}V - \int_V \delta\mathbf{u}^T\dot{\mathbf{f}}\mathrm{d}V - \int_S \delta\mathbf{u}^T\dot{\mathbf{q}}\mathrm{d}S = 0\\ &\Rightarrow \int_V \delta\boldsymbol{\varepsilon}^T\dot{\boldsymbol{\sigma}}'\mathrm{d}V + \int_V \delta\boldsymbol{\varepsilon}^T A_1\dot{u}_a\mathbf{M}\mathrm{d}V + \int_V \delta\boldsymbol{\varepsilon}^T A_2\dot{u}_w\mathbf{M}\mathrm{d}V - \int_V \delta\boldsymbol{\varepsilon}^T A_3\dot{\varepsilon}_v\mathbf{M}\mathrm{d}V\\ &= \int_V \delta\mathbf{u}^T\dot{\mathbf{f}}\mathrm{d}V + \int_S \delta\mathbf{u}^T\dot{\mathbf{q}}\mathrm{d}S\end{aligned} \tag{8}$$

where $\mathbf{q}$ is the surface force. The weak form description of the fluid phase continuity is

$$\begin{aligned}&\int_V \delta u_w\mathrm{div}\mathbf{v}_w^r\mathrm{d}V - \int_V \delta u_w\psi\dot{\varepsilon}_v\mathrm{d}V + \int_V \delta u_w\bar{a}_{11}\dot{u}_w\mathrm{d}V - \int_V \delta u_w a_{12}\dot{u}_a\mathrm{d}V\\ &= \int_V (\nabla\delta u_w)^T\mathbf{v}_w^r\mathrm{d}V + \int_V \delta u_w\psi\dot{\varepsilon}_v\mathrm{d}V\\ &- \int_V \delta u_w\bar{a}_{11}\dot{u}_w\mathrm{d}V + \int_V \delta u_w a_{12}\dot{u}_a\mathrm{d}V - \int_S \delta u_w v_{wn}^r\mathrm{d}S = 0\end{aligned} \tag{9}$$

where v_{wn}^r is the given normal filtration velocity of the fluid phase. The weak form description of the air phase continuity is

$$\begin{aligned}&\int_V \delta u_a\mathrm{div}\mathbf{v}_a^r\mathrm{d}V - \int_V \delta u_a(1-\psi)\dot{\varepsilon}_v\mathrm{d}V + \int_V \delta u_a\bar{a}_{22}\dot{u}_a\mathrm{d}V - \int_V \delta u_a a_{21}\dot{u}_w\mathrm{d}V\\ &= \int_V (\nabla\delta u_a)^T\mathbf{v}_a^r\mathrm{d}V + \int_V \delta u_a(1-\psi)\dot{\varepsilon}_v\mathrm{d}V\\ &- \int_V \delta u_a\bar{a}_{22}\dot{u}_a\mathrm{d}V + \int_V \delta u_a a_{21}\dot{u}_w\mathrm{d}V - \int_S \delta u_a v_{an}^r\mathrm{d}S = 0\end{aligned} \tag{10}$$

where v_{an}^r is the given normal filtration velocity of the air phase. Eqs. (8)–(10) are the weak form description of consolidation of unsaturated soils.