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Research Paper

On the finite element formulation of dynamic two-phase coupled problems

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ABSTRACT

Two finite element approaches are discussed for the analysis of the coupled problems of seepage and deformation of saturated porous media in the presence of an acceleration field varying in time and space (e.g. during an earthquake). The equations governing the two phase problem in dynamic regime are recalled first under assumptions which seem reasonable in the geotechnical context. Then they are cast into a first finite element form without introducing further assumptions with respect to those adopted in deriving them. Subsequently, a simplified formulation is presented which requires a reduced number of nodal variables with respect to the first one. After discussing a time integration scheme, the two approaches are applied to the solution of a benchmark example and some comparative comments are presented on their accuracy and on the required computational effort.

1. Introduction

The problem discussed here is related to the design of deep retaining structures, such as diaphragm walls, in seismic regions. In particular, the case of excavations in granular soils below the water table is considered.

In these conditions the assessment of the effects of earthquakes on the stability and deformation of the structure requires the evaluation of the dynamic effective stresses and pore pressure that the saturated soil exerts on it. Note, in fact, that the relatively high hydraulic conductivity of granular soils rules out the assumption of undrained conditions sometime adopted in engineering practice when dealing with cohesive materials.

In relatively simple cases, e.g. gravity retaining walls, this problem can be tackled through well-established theories such as those originally proposed by Okabe in 1926 [1] and by Mononobe and Matsuo in 1929 [2] for the evaluation of the effective pressure, and by Westergaard in 1933 [3] for estimating the dynamic increase of water pressure. In more complex conditions, however, a coupled dynamic analysis of seepage flow and deformation of the soil skeleton is required.

In quasi static conditions, i.e. under a gravity acceleration field constant in time and space, broadly accepted numerical approaches are available for the numerical analysis of seepage and of the coupled effective stress-flow problem, e.g. [4–6].

When the acceleration field varies with time, e.g. during earthquakes, the analysis of seepage becomes less straightforward since recourse cannot be made anymore to the usual concept of hydraulic head [7,8]. This led, in turn, to various numerical approaches for dynamic coupled problems that involve different assumptions and different sets of independent variables [9–12].

The complex mathematical structure of the dynamic two-phase problem does not permit a straightforward evaluation of the consequences of these assumptions and, hence, makes the choice of the most appropriate numerical approach somewhat controversial.

Here, a previous study concerning the numerical analysis of dynamic seepage [13–15] is extended to the coupled two-phase analysis. This work neglects the possible development of large strains in the soil mass, which was considered in other works recently presented in the literature, see e.g. [16,17].

First, the equations governing the dynamic flow of a liquid within a deformable porous medium are recalled and are coupled with those governing the deformation of its skeleton. Then they are re-written in finite element form. These derivations are presented in some detail to allow the interested reader to follow their various steps.

On these bases two alternative finite element formulations are described. The first one does not introduce further assumptions with respect to those on which the governing equations are based. In this case the nodal variables consist of the displacements for the solid phase and of the relative seepage velocity for the liquid phase.

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Note that other formulations presented in the literature, e.g. [10,12], adopt as independent variables the displacements of both solid and liquid phases.

The second formulation represents a simplified approach which reduces the nodal variables to the displacements of the solid phase only.

An iterative time integration scheme is then outlined for both formulations and is applied in the solution of a test problem. The numerical results suggest some observations on the advantages and drawbacks of the two approaches in term of accuracy and computational effort.

In the following, the problem is approached considering the saturated porous medium equivalent to two superimposed continua, referred to as solid and liquid phases. The two phases have the same volume, which coincides with that of porous medium. This assumption involves the use of equivalent quantities that will be defined subsequently.

A matrix notation and an Eulerian approach with a constant geometry are adopted in this paper. All variables are in general functions of time t. Upper and lower case bold face letters denote matrices and column vectors, respectively. A superposed dot and a superscript T denote time derivative and transpose.

2. Equations governing the seepage flow

The dynamic equations governing the seepage flow of a liquid within a fully saturated porous skeleton are recalled here introducing some assumptions that seem reasonable in the geotechnical context. In particular, a Newtonian pore liquid is considered, referred to in the following as water, with constant deviatoric viscosity and no volumetric viscosity; this liquid has a constant density and its volumetric deformation linearly depends on the pore pressure; isothermal conditions are assumed, thus neglecting the influence of temperature; the fluid flow is laminar.

Let introduce now the following quantities: $\boldsymbol{v}_W$ is the vector collecting the Cartesian components of the velocity of the water particles; $\boldsymbol{v}$ represents the discharge velocity of flow in Darcy sense, which pertains to the liquid phase; $\dot{\boldsymbol{u}}$ is the velocity of the solid phase, which coincides with that of the soil skeleton; $\boldsymbol{w}_W$ is the relative velocity of the water particles with respect to the skeleton and $\boldsymbol{w}$ is the relative discharge velocity. The following relationships hold between these variables.

$$\boldsymbol{w}_W = \boldsymbol{v}_W - \dot{\boldsymbol{u}}, \quad \boldsymbol{w} = \boldsymbol{v} - \dot{\boldsymbol{u}} \tag{1a, b}$$

The relative discharge velocity $\boldsymbol{w}$ depends on the relative velocity of the water particles $\boldsymbol{w}_W$ through the matrix $\boldsymbol{N}_A$ of the area porosities.

$$\boldsymbol{w} = \boldsymbol{N}_A \boldsymbol{w}_W \tag{2}$$

If the principal directions of porosity coincide with the Cartesian axes, $\boldsymbol{N}_A$ is a 3×3 diagonal matrix the entries of which n_{Ax}, n_{Ay} and n_{Az} are the ratios between the area of pores and the total area of the sections normal to the reference axes.

The difficulties met in determining the area porosities suggest using the volume (or effective) porosity n, which represents the ratio between the volume of voids and the total volume of a soil element, and that can be seen as the average value of the area porosities [7]. Consequently, Eq. (2) becomes.

$$\boldsymbol{w} = n\boldsymbol{w}_W \tag{3}$$

2.1. Equation of compatibility

The equation of compatibility of the liquid phase relates its strain rates, collected in vector $\dot{\boldsymbol{\varepsilon}}_L$, to the discharge velocity $\boldsymbol{v}$ through the same 6x3 differential operator $\boldsymbol{B}$ (see the List of Symbols) that governs the strain–displacement relationship for solids.

$$\dot{\boldsymbol{\varepsilon}}_L = \boldsymbol{B}\boldsymbol{v} \tag{4}$$

Considering Eq. (1b), the relationship between the strain rate of the liquid phase $\dot{\boldsymbol{\varepsilon}}_L$, the relative discharge velocity $\boldsymbol{w}$ and the skeleton velocity $\dot{\boldsymbol{u}}$ is.

$$\dot{\boldsymbol{\varepsilon}}_L = \boldsymbol{B}\boldsymbol{w} + \boldsymbol{B}\dot{\boldsymbol{u}} \tag{5}$$

2.2. Shear stress-shear strain rate relationship

The stresses $\boldsymbol{\sigma}_L$ and the strain rates $\dot{\boldsymbol{\varepsilon}}_L$ of the liquid phase are expressed through the following quantities,

$$p = \frac{1}{3}\boldsymbol{m}^T\boldsymbol{\sigma}_L; \quad \boldsymbol{\tau}_L = \boldsymbol{\sigma}_L - \boldsymbol{m}p = \left(\boldsymbol{I} - \frac{1}{3}\boldsymbol{m}\boldsymbol{m}^T\right)\boldsymbol{\sigma}_L \tag{6a, b}$$

$$\dot{\varepsilon}_{L,vol} = \boldsymbol{m}^T\dot{\boldsymbol{\varepsilon}}_L; \quad \dot{\boldsymbol{e}}_L = \dot{\boldsymbol{\varepsilon}}_L - \frac{1}{3}\boldsymbol{m}\dot{\varepsilon}_{L,vol} = \left(\boldsymbol{I} - \frac{1}{3}\boldsymbol{m}\boldsymbol{m}^T\right)\dot{\boldsymbol{\varepsilon}}_L \tag{7a, b}$$

where p is the pore pressure (positive if tensile), which coincides with the volumetric part of the stresses $\boldsymbol{\sigma}_L$; $\boldsymbol{\tau}_L$ is the deviatoric stress vector; $\dot{\varepsilon}_{L,vol}$ and $\dot{\boldsymbol{e}}_L$ are the volumetric and deviatoric strain rates; $\boldsymbol{I}$ is the identity matrix and $\boldsymbol{m}$ is a 6 component vector the entries of which are equal to 1 if they correspond to normal strains/stresses, otherwise they vanish.

In a Newtonian liquid, a linear relationship holds between stresses and strain rates which is formally analogous to that relating stresses and strains for a linearly elastic solid. In the case of solids the law depends on bulk and shear elastic moduli; in the case of liquids on bulk and shear viscosities. Since the bulk viscosity is neglected in the present context, only the deviatoric part of the law remains.

$$\boldsymbol{\tau}_L = \mu_L \boldsymbol{I}_0 \dot{\boldsymbol{e}}_L \tag{8}$$

Here μ_L is the deviatoric viscosity of the liquid phase and $\boldsymbol{I}_0$ is a 6×6 diagonal matrix with entries equal to 2 if they correspond to normal strains, otherwise they are equal to 1.

Eqs. (8), (7) and (5) lead to the following $\boldsymbol{\tau}_L - \boldsymbol{w}$ relationship,

$$\boldsymbol{\tau}_L = \mu_L \boldsymbol{I}_1 (\boldsymbol{B}\boldsymbol{w} + \boldsymbol{B}\dot{\boldsymbol{u}}) \tag{9}$$

where

$$\boldsymbol{I}_1 = \boldsymbol{I}_0 - \frac{1}{3}\boldsymbol{I}_0\boldsymbol{m}\boldsymbol{m}^T = \boldsymbol{I}_0 - \frac{2}{3}\boldsymbol{m}\boldsymbol{m}^T \tag{10}$$

Substituting Eq. (9) into Eq. (6b) one obtains.

$$\boldsymbol{\sigma}_L = \mu_L \boldsymbol{I}_1 (\boldsymbol{B}\boldsymbol{w} + \boldsymbol{B}\dot{\boldsymbol{u}}) + \boldsymbol{m}p \tag{11}$$

2.3. Conservation of the mass of liquid

If internal sources are neglected, the conservation condition requires that the liquid phase mass, $\dot{m}_1$, accumulated in a unit volume in a unit time coincides with the difference, $\dot{m}_2$, between the rates of mass entering and leaving it.

$$\dot{m}_1 = \dot{m}_2 \tag{12}$$

The rate of mass accumulation, $\dot{m}_1$, consists of four contributions. The first one depends on the volumetric strain rate of the skeleton, $\dot{\varepsilon}_{S,vol}$, which in turn is a function of its displacement rate $\dot{\boldsymbol{u}}$.

$$\dot{\varepsilon}_{S,vol} = \boldsymbol{m}^T(\boldsymbol{B}\dot{\boldsymbol{u}}) \tag{13}$$

Since positive volume strains correspond to a volume increase, a positive value of $\dot{\varepsilon}_{S,vol}$ involves an increase of the liquid mass within the volume.