Computers and Geotechnics 62 (2014) 1–10

Contents lists available at ScienceDirect

Computers and Geotechnics

journal homepage: www.elsevier.com/locate/compgeo

Consolidation analysis of non-homogeneous soil by the weak form quadrature element method

Shuai Yuan, Hongzhi Zhong *

Department of Civil Engineering, Tsinghua University, Beijing 100084, PR China

ARTICLE INFO

Article history:
Received 16 April 2014
Received in revised form 24 June 2014
Accepted 25 June 2014
Available online 14 July 2014

Keywords:
Weak form quadrature element method
Non-homogeneous soil
Biot's theory
Consolidation

ABSTRACT

A weak form quadrature element formulation for consolidation analysis of non-homogeneous saturated soil is established based on Biot's theory. Numerical examples are given, and the results are compared with the analytical solutions available or those from the commercial finite element software ABAQUS, demonstrating accuracy and rapid convergence of quadrature element solutions. The disparity between various treatments of non-homogenous soil is discussed, and the effectiveness and advantages of the quadrature element formulation in consolidation analysis of non-homogeneous soil are demonstrated.

1. Introduction

Consolidation is a time-dependent process involving the dissipation of porous fluid pressure and the deformation of the soil skeleton. This process is one of the most important issues in geotechnical engineering. Attempts to address geotechnical problems involving multi-phase materials can be traced back to the 18th century [1]. Terzaghi presented the famous one-dimensional theory of soil consolidation in 1925 [2]. Biot proposed the three-dimensional theory of consolidation [3], which has become widely applicable in geotechnical engineering and has been extended to irreversible thermodynamics of polyphasic reactive open continua [4,5]. Biot's theory consists of the continuity of porous fluid flow and differential equilibrium equations of soil that takes into account the coupling of the dissipation of porous fluid pressure and deformation of the soil skeleton. Although the theory has been shown to be very effectual, closed form solutions are not available for complex practical problems. Thus far, a number of numerical methods have been employed to address consolidation problems, such as finite element methods [6–9], finite difference methods [10], boundary element methods [11–15], meshfree methods [16,17] and various other methods [18]. To find numerical solutions for a consolidation problem, plane strain or an axisymmetric condition is often assumed to reduce the number of degrees of freedom in the numerical model.

Soil is inhomogeneous across its depth due to the formation process, yet it is often taken as a uniform media in conventional consolidation analysis for simplicity. Unfortunately, large deviations will be induced due to the simplification of inhomogeneity when the variation is large. To account for the non-homogenous effect of soil, substantial research has been conducted that has placed soil in two different categories: soil with continuously varying parameters and multi-layered soil. Soils in the first category are treated as a media with continuously varying properties (e.g., [19–22]). Soils in the second category are divided into a number of layers, where homogeneous or non-homogeneous properties are assumed in each layer [23–26].

Despite these research efforts, much of the research work in either category has been conducted based on one-dimensional consolidation analysis, which is not sufficient when external loads are not uniformly distributed or properties of simulated soils vary in the horizontal direction. In the present paper, the weak form quadrature element method (QEM) is employed to study the behaviour of two-dimensional non-homogenous saturated soil based on Biot's theory. The QEM is an efficient numerical tool that has been applied to various structural problems [27–30]. In the QEM, integrals in the weak form description of a problem are first evaluated by numerical integration, and then the derivatives at an integration point are represented by the differential quadrature analogue. The essence of the differential quadrature analogue is that a derivative is approximated by the linear weighted sum of variables at sampling points. Thus, algebraic equations are established from which all the function variables are obtained. The QEM has the characteristic of global approximation and enjoys

* Corresponding author. Tel./fax: +86 10 62781891.
E-mail address: hzz@tsinghua.edu.cn (H. Zhong).

http://dx.doi.org/10.1016/j.compgeo.2014.06.012
0266-352X/

rapid convergence. The pre-process in quadrature element analysis is fairly simple, as one quadrature element often suffices for a convex quadrilateral problem domain (any inner angle of the quadrilateral is less than 180°) in the analysis of two-dimensional problems, for instance. For one-dimensional non-homogeneous soil in either category, soil with continuously varying properties or multi-layered soil, the division of soil is minimised in quadrature element analysis because one element is enough for one layer. The inhomogeneity of soil is accounted for with no additional effort in a quadrature element formulation, especially for soils with complex inhomogeneous properties such as an exponentially or hyperbolically varying modulus. Numerical implementation of the quadrature element formulation is rather straightforward because it requires less upfront data preparation. In this paper, a quadrature element formulation for solution of the coupled partial differential equations of Biot's theory for non-homogeneous soil is established. The weak-form governing equations are obtained by the principle of virtual work. Numerical examples of soil consolidation are presented, and the results are compared with analytical solutions and those from the finite element software ABAQUS. Excellent agreement is reached, demonstrating the robustness and accuracy of the present formulation.

2. Formulation

In the present work, boldfaced letters indicate matrices and vectors, and boldfaced letters with subscripts *i*,*j* stand for their values at node (*i*,*j*). The gravity force and compressibility of soil grains and porous fluids are neglected. The pore pressure p, the stress $\boldsymbol{\sigma}$ and the strain $\boldsymbol{\varepsilon}$ are all positive for compression. In Biot's theory, the differential equations consist of equilibrium conditions of soil and continuity of porous fluid flow, i.e.,

$$\mathrm{div}\boldsymbol{\sigma} = 0 \tag{1}$$

$$\mathrm{div}\mathbf{v} - \dot{\varepsilon}_v = 0 \tag{2}$$

where the single dot stands for the first order derivative with respect to time, ε_v the volumetric strain and $\mathbf{v}$ is the filtration velocity of pore fluid. It is noted that the total stress can be given in terms of Terzaghi's effective stress, i.e.,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' + \mathbf{M}p \tag{3}$$

where $\boldsymbol{\sigma}'$ is the effective stress, and the identity vectors are $\mathbf{M} = \{1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0\}^T$ and $\mathbf{M} = \{1 \quad 1 \quad 0\}^T$ for three-dimensional and two-dimensional problems, respectively. Hence, Darcy's law and the constitutive law of the soil skeleton are given as

$$\mathbf{v} = -\frac{\mathbf{K}}{\mu}\nabla p, \tag{4}$$

$$\boldsymbol{\sigma}' = \mathbf{E}\boldsymbol{\varepsilon}, \tag{5}$$

where $\mathbf{K}$ is the permeability matrix, μ is the dynamic viscosity of the porous fluid, and $\mathbf{E}$ is the elasticity matrix of the soil skeleton. The weak form description of these governing differential equations is obtained by the principle of virtual work, i.e.,

$$\int_V \delta\boldsymbol{\varepsilon}^T\boldsymbol{\sigma}dV - \int_S \delta\mathbf{d}^T\mathbf{T}dS = 0 \tag{6}$$

or

$$\int_V \delta\boldsymbol{\varepsilon}^T\mathbf{E}\boldsymbol{\varepsilon}dV + \int_V \delta\boldsymbol{\varepsilon}^T\mathbf{M}pdV - \int_S \delta\mathbf{d}^T\mathbf{T}dS = 0 \tag{7}$$

where $\mathbf{d}$ is the displacement vector, and $\mathbf{T}$ is the surface force vector. Similarly, using the principle of virtual work, the continuity equation can be written as

$$\int_V \delta p\mathrm{div}\mathbf{v}dV - \int_V \delta p\dot{\varepsilon}_v dV = 0 \tag{8}$$

Integration by parts of the first volume integral in Eq. (8) and the incorporation of $\dot{\varepsilon}_v = \mathbf{M}^T\dot{\boldsymbol{\varepsilon}}$ into Eq. (8) give

$$-\int_V \nabla(\delta p)^T\mathbf{v}dV - \int_V \delta p\mathbf{M}^T\dot{\boldsymbol{\varepsilon}}dV + \int_S \delta p v_n dS = 0 \tag{9}$$

where v_n is the normal filtration velocity across the boundary. The introduction of Darcy's law into Eq. (9) yields

$$\int_V \nabla(\delta p)^T\frac{\mathbf{K}}{\mu}\nabla p dV - \int_V \delta p\mathbf{M}^T\dot{\boldsymbol{\varepsilon}}dV + \int_S \delta p v_n dS = 0 \tag{10}$$

Eqs. (7) and (10) are the constituents of the weak form description of the saturated soil based on Biot's theory.

3. Weak form quadrature element formulation

Weak form quadrature element analysis sets out from the approximation of the integrals of the weak form description of the problem with numerical integration. For instance, a two-dimensional problem domain is first discretised into a few quadrilateral subdomains (quadrature elements) where numerical integration can be carried out as illustrated in Fig. 1. Then, every subdomain is transformed into the standard computational domain, i.e.,

$$\begin{cases} x = x(\xi,\eta) \\ y = y(\xi,\eta) \end{cases} \quad -1 \leqslant \xi, \quad \eta \leqslant 1 \tag{11}$$

where x and y are coordinates in the physical domain and ξ and η are coordinates in the standard domain. With the chain rule of differentiation,

$$\begin{Bmatrix} \frac{\partial}{\partial\xi} \\ \frac{\partial}{\partial\eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial\xi} & \frac{\partial y}{\partial\xi} \\ \frac{\partial x}{\partial\eta} & \frac{\partial y}{\partial\eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \mathbf{J}\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} \tag{12}$$

and

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1}\begin{Bmatrix} \frac{\partial}{\partial\xi} \\ \frac{\partial}{\partial\eta} \end{Bmatrix} \tag{13}$$

where $\mathbf{J}$ is the Jacobian matrix. The introduction of numerical integration into Eq. (7) yields

$$\sum_{i=1}^{N_\xi}\sum_{j=1}^{N_\eta} W_iW_j\delta\boldsymbol{\varepsilon}_{ij}^T\mathbf{E}_{ij}\boldsymbol{\varepsilon}_{ij}|\mathbf{J}|_{ij} + \sum_{i=1}^{N_\xi}\sum_{j=1}^{N_\eta} W_iW_j\delta\boldsymbol{\varepsilon}_{ij}^T\mathbf{M}p_{ij}|\mathbf{J}|_{ij} - \sum_{j=1}^{N_S}\sum_{i=1}^{N_j} W_i\delta\mathbf{d}_i^T\mathbf{T}_i|\bar{\mathbf{J}}|_i = 0 \tag{14}$$

where W_i and W_j are the weighting coefficients of the integration scheme, N_ξ and N_η are the number of integration points in the two directions of the element, N_S is the number of boundary surfaces where external surface loads are applied, and N_j is the number of integration points on the jth surface. $|\mathbf{J}|$ and $|\bar{\mathbf{J}}|$ are the Jacobians for the element domain and its boundary. For a quadrilateral quadrature element, Lobatto quadrature is often chosen for efficiency.

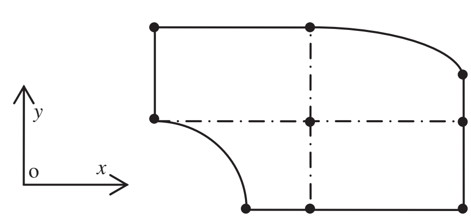

Fig. 1. Domain division in the weak form quadrature element method.