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Isogeometric analysis for unsaturated flow problems

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ABSTRACT

Unsaturated flow problems in porous media often described by Richards' equation are of great importance in many engineering applications. In this contribution, we propose a new numerical flow approach based on isogeometric analysis (IGA) for modeling the unsaturated flow problems. The non-uniform rational B-spline (NURBS) basis is utilized for spatial discretization whereas the stable implicit backward Euler method for time discretization. The nonlinear Richards' equation is iteratively solved with the aid of the Newton–Raphson scheme. Owing to some desirable features of an efficient numerical flow approach, major advantages of the present formulation involve: (a) numerical oscillation at the wetting front can be avoided or facilitated, simply by using either an *h*-refinement or a lumped mass matrix technique; (b) higher-order exactness can be obtained due to the nature of the IGA features; (c) the approach is straightforward to implement and it does not need any transformation, e.g., Kirchhoff transformation or filter algorithm; and (d) in contrast to the Picard iteration scheme, which forms linear convergences, the proposed approach can however yield quadratic convergences by using the Newton–Raphson method for solving resultant nonlinear equations. Numerical model validation is analyzed by solving a three-dimensional unsaturated flow problem in soil, and its derived results are verified against analytical solutions. Numerical applications are then studied by considering three extensive examples with simple and complex configurations to further show the accuracy and applicability of the present IGA.

1. Introduction

Landslides induced by rainfall, dike failure caused by the infiltration of rain and flood, and contaminant transportation into soil, etc. are relevant to the flow in unsaturated porous media, and the flow is often governed by Richard's equation [1], which is the combination of Darcy's law and the continuity equation.

Due to the strong non-linearity of the Richard's equation, only few unsaturated flow problems with simple initial and boundary conditions can be analytically solved [2–8]. For general unsaturated flow problems, numerical methods are much more effective. In the past decades several numerical models have been developed to solve unsaturated flow problems. In general, the finite difference method [9–11], the finite element method [12–18], the flux-concentration [19,20], the finite volume method [21,22] and the meshless method [23], etc. are used for spatial discretization while the finite difference method for time discretization, and the discretized nonlinear Richards' equation is then solved iteratively. Recently, a computational model of unsaturated flow in porous media based on a phase-field formulation is presented in [24] by extending the Richard's equation to predict the instability and capture the key features of gravity fingering quantitatively.

In spite of the success and the great variety of existing numerical methods for the solution of the unsaturated flow problems, there is still a growing interest in the development of new advanced methods. In recent years the isogeometric analysis (IGA) [25] is becoming popular due to a number of advanced features including the exactness of reproducing the geometry, higher-order continuity, simple mesh refinement, and avoiding the traditional mesh generation procedure. The IGA has been successfully applied to implement many engineering problems including high-performance [26], plates [27–29], incompressibility [30], electromagnetics [31] and phase fields [32,33], etc. Recently, the IGA is also developed for poroelasticity, using Biot's model for fully saturated condition [34].

The objective of the present work is to propose a new numerical flow approach in the framework of the IGA for modeling the

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unsaturated flow problems. The present formulation utilizes the non-uniform rational B-spline (NURBS) basis for spatial discretization whereas the stable implicit backward Euler method for time discretization. We apply the Newton–Raphson technique to iteratively solve the nonlinear Richards' equation. The key advantages of the developed method can be highlighted as follows:

(a) Numerical oscillation at the wetting front can be avoided or facilitated by using *h*-refinement or lumped mass matrix techniques.
(b) Higher-order exactness can be obtained because of the nature features of the IGA method.
(c) The numerical implementation of the proposed approach is straightforward and it avoids Kirchhoff transformation or filter algorithm.
(d) The quadratic convergence can be obtained instead of the linear one as some existing references using the Picard iteration.

We verify the accuracy of the proposed model through a numerical model validation by dealing with a three-dimensional unsaturated flow problem in soil, in which L2 norm error is explored. The accuracy and applicability of the method is further illustrated through three other numerical examples of unsaturated flow problems. We respectively analyze the infiltration into New Mexico soil [11], a nonlinear traveling shock [23,35] and a semi-circular furrow into homogeneous soil in three-dimension [36]. Obviously, the obtained results are verified with respect to reference solutions to show the accuracy and applicability of the present approach.

The rest of the paper is organized as follows. After the introduction, Section 2 briefly presents basic equations for unsaturated flow in porous media. The IGA for unsaturated flow is then described in Section 3. Key steps of numerical solution procedure are presented in Section 4. Numerical validation is presented in Section 5 while numerical applications and discussions using the proposed IGA are presented in Sections 6 and 7 respectively. Some conclusions drawn from the study are presented in Section 8.

2. Basic equations for unsaturated flow

The fluid movement in unsaturated and non-swelling porous media is governed by the Richards' equation. Based on the main variable considered, i.e., the pressure head *h*, or the moisture content θ, the Richards' equation can be written in two different forms as follows: the head-based Richards equation is

$$C(h)\frac{\partial h}{\partial t} = \nabla \cdot (\mathbf{K}(h)\nabla h) + \frac{\partial \mathbf{K}(h)}{\partial z} \tag{1}$$

and the moisture-based Richards equation is

$$\frac{\partial \theta}{\partial t} = \nabla \cdot (D(\theta)\nabla \theta) + \frac{\partial \mathbf{K}}{\partial z} \tag{2}$$

where $C(h) = \frac{d\theta}{dh}$ is the specific moisture capacity function, $\mathbf{K}(h)$ is the unsaturated hydraulic conductivity tensor, $\mathbf{D}(\theta) = \frac{\mathbf{K}(\theta)}{C(\theta)}$ is the unsaturated diffusivity, *z* denotes the vertical co-ordinate (positive downward) and ∇ is the gradient operator. The unsaturated hydraulic conductivity can be determined from the saturated one by

$$\mathbf{K} = K_r(\theta)\mathbf{K}_s \tag{3}$$

where $\mathbf{K}_s$ is a tensor of the fully saturated hydraulic conductivity while K_r is a function of the moisture content θ, termed as the relative hydraulic conductivity. Note that the hydraulic conductivity in general could be anisotropic, however it is considered to be isotropic in this work, i.e., $\mathbf{K}_s = K_s\mathbf{I}$, with $\mathbf{I}$ being the second order identity tensor.

Each form of the Richards' equations has its own advantages and disadvantages. The head-based form is applicable to both saturated and unsaturated conditions, but it generally yields poor performance because of the large mass balance error and erroneous estimates of the infiltration depth. Obviously, further specific techniques are required in the head-based form to minimize the resulting error [11,37]. In contrast, the moisture-based form dominates over the former one as it is automatically mass conservative without any additional technique, but the applicability of the moisture-based form is restricted to unsaturated conditions. There also exists in the literatures another form of the Richards' equation, i.e., the mixed form

$$\frac{\partial \theta}{\partial t} = \nabla \cdot (\mathbf{K}(h)\nabla h) + \frac{\partial \mathbf{K}(h)}{\partial z} \tag{4}$$

In order to solve the mixed form of the Richards' equation, a primary variable has to be chosen in the beginning. With the choice of either the pressure head or the moisture content as its primary variable, many properties described above are encountered again.

In this study, we focus on the solution for the unsaturated flows. Therefore the moisture-based Richards' equation is an appropriate choice, and will be presented in the subsequent sections.

3. IGA formulation for unsaturated flow

3.1. The NURBS basis functions

In this section, a brief summary of some technical features of the non-uniform rational B-spline (NURBS) is presented. A more detailed description can be found in [38]. A NURBS curve, $\widehat{C}(\xi)$, of order *p* is the linear combination of the NURBS basis functions, in which the coefficients are a given set of the control points

$$\hat{\mathbf{C}}(\xi) = \sum_{i}^{n} R_{i,p}(\xi)\boldsymbol{P}_i \tag{5}$$

where *n* is the number of the control points, $\boldsymbol{P}_i$ is the control point coordinates, and $R_{i,p}(\xi)$ is the univariate NURBS basis functions determined by

$$R_{i,p}(\xi) = \frac{N_{i,p}(\xi)w_i}{\sum_{i=1}^{n} N_{i,p}(\xi)w_i} \tag{6}$$

where w_i is the non-negative weight assigned for the *i*th control point, and $N_{i,p}(\xi)$ are the B-spline basis functions of order *p*.

To construct a set of *n* B-spline basis functions of order *p*, a knot vector $\boldsymbol{k}(\xi)$ with non-decreasing sequence of the real numbers in a parametric space, $\xi \in [0, 1]$, is defined as

$$\boldsymbol{k}(\xi) = \{\xi_1 = 0, \ldots, \xi_i, \ldots, \xi_{n+p+1} = 1\} \tag{7}$$

where ξ_i is called the *i*th knot.

A knot vector is said to be open if the knots are repeated $p + 1$ times at the start and end of the vector. For the analysis purposes, the open knot vectors are generally used to take advantage of the Kronecker-delta property at the boundary points, allowing direct application of the essential boundary conditions at these points. Given a knot vector, the univariate B-spline basis function, $N_{i,p}(\xi)$, can be constructed recursively following the Cox-de Boor formulation

$$N_{i,0}(\xi) = \begin{cases} 1 & \xi_i \leqslant \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases} \quad \text{for } p = 0 \tag{8a}$$

and

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i}N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}}N_{i+1,p-1}(\xi) \quad \text{for } p \geqslant 1 \tag{8b}$$

The B-spline basis functions constructed from an open knot vector have the interpolation property at both ends of the parametric