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Evaluate the performance of an accelerated initial stiffness method in three dimensional finite element analysis

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ABSTRACT

Newton's method is a commonly used algorithm for elasto-plastic finite element analysis and has three common variations: the full Newton–Raphson method, the modified Newton–Raphson method and the initial stiffness method. The Newton–Raphson methods can converge to the solution in a small number of iterations when the system is stable; however, the methods can be quite computationally expensive in some types of problems, for example where the tangent stiffness matrix is unsymmetric or the plasticity is highly localized. The initial stiffness method is robust in those cases but requires a larger number of iterations. This prompted the formulation of many acceleration techniques in literature. In this paper, those techniques will be briefly discussed. This will be followed by the development of a modified acceleration technique for the initial stiffness method. The performance of the modified accelerated initial stiffness method will be examined in elasto-plastic analyses, using both direct and iterative matrix solvers. The results will be compared – in terms of the required number of iterations and the computation time – with an existing accelerated initial stiffness method, the non-accelerated initial stiffness method and the Newton–Raphson tangent stiffness method.

1. Introduction

Newton's method is a commonly used computational algorithm for elasto-plastic finite element analysis. In the general form, an unbalanced force is applied to the system of equations and the resulting displacement is calculated. This process is repeated until equilibrium is achieved within a specified tolerance. Three common variations of this method include the Newton–Raphson method, the modified Newton–Raphson method and the initial stiffness method. The full Newton–Raphson method uses the tangent stiffness matrix which needs to be formed and factorized for each iteration. The method often provides fast convergence but is quite computationally expensive for complex constitutive laws (i.e. if the non-associated plastic flow rule is employed). In the modified Newton–Raphson method, the tangent stiffness matrix is recalculated only once at the beginning of each load step, or when the convergence rate drops below a pre-set rate. This is less computationally expensive, but results in a slower convergence rate compared to the full Newton–Raphson method [1].

In the initial stiffness method, the global stiffness matrix is constructed once at the beginning of the calculation and is kept constant throughout the analysis. This method faces one major shortcoming as it requires a greater number of iterations to converge. This is most apparent when dealing with systems with excessive plasticity; in these cases the initial stiffness method requires a greater number of iterations than the Newton–Raphson methods.

Attempts to modify the Newton–Raphson method have been unsuccessful, as the modified methods require an excessive number of iterations to achieve convergence. Simo and Taylor [2] introduced the consistent tangent operator to maintain the quadratic rate of convergence from the original formulation of the Newton–Raphson method. However, the complexity in formulating a consistent tangent operator only provided more justification for pursuing acceleration techniques for the initial stiffness method.

Acceleration techniques have been proposed to accelerate the rate of convergence for the initial stiffness method in an attempt to make it more favorable. Aiken's method [3] assumes the error decays exponentially, but it is unreliable for computer applications since it has a tendency to predict infinite results when the denominator approaches or becomes zero. Boyle and Jennings [4] proposed a modified Aitken acceleration technique, which rectified the problem and showed considerable promise, as convergence was achieved faster and with a smaller number of iterations. Nayak and Zienkiewicz [5] proposed a similar acceleration scheme in which

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each degree of freedom is accelerated individually. This procedure was reported to be unreliable in practice by Sloan et al. [6]. Crisfield [7] developed a single parameter acceleration scheme and employed the line search technique to stabilize the iterations. This method was, however, designed for the modified Newton–Raphson method. The efficiency of this technique on the initial stiffness method has not been reported in literature.

Recently, an important modification to the initial stiffness method was made by researchers including Thomas [8] and Chen [9]. They proposed single-parameter acceleration techniques which benefited from being simple yet robust. The procedure proposed by Thomas [8] uses the least-squares minimization of the displacement. The method proposed by Chen [9] is based on the least-squares minimization of the out-of-balance load at the end of the current iteration. Sloan et al. [6] proposed a modified Thomas acceleration scheme upon reviewing Chen and Thomas's formulations. In this paper, this will be referred to as the Sloan Acceleration method. The normal procedure requires one back substitution and one unbalanced force evaluation for each iteration, while those formulations require either two back substitutions, two unbalanced force evaluations or both in order to determine the single parameter.

To the best of the author's knowledge, there have not been any reports on the performance of accelerated initial stiffness methods in 3D analyses. It would be interesting to examine if the performance of these methods would be preserved in such a simulation. Since solving 3D problems is more computationally intensive than 2D analysis, the limitations of direct matrix solvers for the analysis of three-dimensional problems using finite element analysis are well known due to the memory storage requirements for the inversion procedure. The available alternative approach is to utilize iterative solution strategies for solving the equilibrium equations. Using iterative algorithms in the finite element method allows problems to be solved with less memory requirements leading to significant time savings compared to out-of-core direct solvers. Unlike direct matrix solvers, the solution of a system of linear equations obtained using the iterative solvers is normally within some tolerance, rather than an exact solution of the equations. The advantages of the accelerated initial stiffness method must also be investigated in simulations using iterative matrix solvers, in addition to simulations using direct matrix solvers.

This paper proposes an acceleration technique for the initial stiffness method using a single parameter. No significant workload is needed to determine the parameter. The formulation was developed and implemented in the 3D finite element program RS3 of Rocscience Inc. [10] and various constitutive models were used to evaluate its performance. Several numerical examples were solved in order to evaluate the performance of the algorithm using both direct and iterative matrix solvers.

2. Formulation of the accelerated initial stiffness method

In the displacement finite element method, the global system of equations to be solved for each iteration can be written as

$$\Delta U^i = K^{-1} R^{i-1} \tag{1}$$

where K is the stiffness matrix, R^{i-1} is the residual force for the previous iteration and ΔU^i is the change in displacement for the current iteration. The residual force, R^{i-1}, is calculated as the difference between the external force and the internal force at the current iteration. The internal force at iteration i is commonly obtained by the relation

$$F^i_{int} = \int B^T \sigma^i dV \tag{2}$$

where B is the strain displacement matrix and σ^i are stresses.

The total displacement U^i at ith iteration is then calculated as

$$U^i = U^{i-1} + \Delta U^i \tag{3}$$

The total displacement, U^i, is the summation of the displacement at the previous iteration, U^{i-1}, and the change in displacement calculated using (1), ΔU^i.

Since the stiffness matrix K depends on the displacements U, it does not remain constant during the iteration procedure. In order to estimate the global stiffness matrix more accurately and thus reduce the number of iterations, the stiffness matrix is recalculated for each iteration in the standard Newton–Raphson method or at the beginning of the load step in the modified Newton–Raphson method. The global stiffness matrix is the tangent stiffness matrix which is calculated as

$$K_{ep} = \int B^T D_{ep} B dV \tag{4}$$

where D_{ep} is the elasto-plastic stress–strain matrix, and B is the strain displacement matrix. However, the workload associated with assembling and factorizing the global stiffness matrix at each iteration may be extensive. Moreover, using the non-associated plastic flow theory normally results in a non-symmetric global stiffness matrix which increases the computational requirements.

On the other hand, the initial stiffness scheme only requires the global elastic stiffness matrix to be formed and factorized once at the beginning of the simulation. In this method, the global stiffness matrix is calculated as

$$K_e = \int B^T D_e B dV \tag{5}$$

where D_e is the elastic stress–strain matrix, and B is the strain displacement matrix.

2.1. Single parameter accelerated initial stiffness scheme

2.1.1. The Sloan scheme

In the acceleration scheme of Sloan et al. [6], the displacement is updated as follows

$$U^{i+1} = U^i + \alpha^i \Delta U^{i+1}_e + K_e^{-1}\left\{R^i - \alpha^i K_{ep} \Delta U^i_e\right\} \tag{6}$$

Using the approximation

$$R^i - \alpha^i K_{ep} \Delta U^i_e \approx R\left(U^i + \alpha^i \Delta U^{i+1}_e\right) \tag{7}$$

The new update may be written as

$$U^{i+1} = U^i + \alpha^i \Delta U^{i+1}_e + \Delta \widetilde{U}^{i+1}_e \tag{8}$$

where

$$\Delta \widetilde{U}^{i+1}_e \approx K_e^{-1}\left\{R\left(U^i + \alpha^i \Delta U^{i+1}_e\right)\right\} \tag{9}$$

In the Sloan method [6], the procedure is started by assuming $\alpha^0 = 1$ at the beginning of the calculation of each load step. After the displacements have been updated using Eq. (8) the acceleration factor for the next iteration, α^{i+1}, is found by performing a least-square fit so that

$$\alpha^{i+1} \Delta U^{i+1}_e \approx \alpha^{i-1} \Delta U^i_e + \Delta \widetilde{U}^i_e \tag{10}$$

This lead to the acceleration factor

$$\alpha^{i+1} = \alpha^i + \frac{\left\{\Delta U^{i+1}_e\right\}^T \left\{\Delta \widetilde{U}^{i+1}_e\right\}}{\left\{\Delta U^{i+1}_e\right\}^T \left\{\Delta U^{i+1}_e\right\}} \tag{11}$$

The algorithm needs two back substitution and two unbalanced force evaluations for each iteration.