Engineering Structures 124 (2016) 344–360

Contents lists available at ScienceDirect

Engineering Structures

journal homepage: www.elsevier.com/locate/engstruct

Semi-implicit finite strain constitutive integration and mixed strain/stress control based on intermediate configurations

P. Areias [a,e,*], T. Rabczuk [b], J. César de Sá [c], J.L. Alves [d]

[a] Department of Physics, University of Évora, Colégio Luís António Verney, Rua Romão Ramalho, 59, 7002-554 Évora, Portugal
[b] Institute of Structural Mechanics, Bauhaus-University Weimar, Marienstraße 15, 99423 Weimar, Germany
[c] Mechanical Engineering Department, Faculty of Engineering, University of Porto, Rua Dr. Roberto Frias, s/n, 4200-465 Porto, Portugal
[d] Mechanical Engineering Department, School of Engineering, University of Minho, Campus de Azurém, 4804-533 Guimarães Portugal
[e] ICIST/CERIS, Instituto Superior Técnico, Lisboa, Portugal

ARTICLE INFO

Article history:
Received 22 December 2015
Revised 21 June 2016
Accepted 22 June 2016

Keywords:
Constitutive integration
Newton methods
Elasto-plasticity
Finite strains

ABSTRACT

A new semi-implicit stress integration algorithm for finite strain plasticity (compatible with hyperelasticity) is introduced. Its most distinctive feature is the use of different parameterizations of equilibrium and reference configurations. Rotation terms (nonlinear trigonometric functions) are integrated *explicitly* and correspond to a change in the reference configuration. In contrast, *relative* Green–Lagrange strains (which are quadratic in terms of displacements) represent the equilibrium configuration *implicitly*. In addition, the adequacy of several objective stress rates in the semi-implicit context is studied. We parametrize both reference and equilibrium configurations, in contrast with the so-called objective stress integration algorithms which use coinciding configurations. A single constitutive framework provides quantities needed by common discretization schemes. This is computationally convenient and robust, as all elements only need to provide pre-established quantities irrespectively of the constitutive model. In this work, mixed strain/stress control is used, as well as our smoothing algorithm for the complementarity condition. Exceptional time-step robustness is achieved in elasto-plastic problems: often fewer than one-tenth of the typical number of time increments can be used with a quantifiable effect in accuracy. The proposed algorithm is general: all hyperelastic models and all classical elasto-plastic models can be employed. Plane-stress, Shell and 3D examples are used to illustrate the new algorithm. Both isotropic and anisotropic behavior is presented in elasto-plastic and hyperelastic examples.

1. Introduction

In retrospective, the implementation of multiple constitutive laws and multiple element formulations limits the choice in terms of finite strain constitutive integration. Distinct constitutive laws should not require distinct finite element implementations and mixed finite element formulations should be able to use any constitutive law implementation. In structural elements, this decoupling was the leitmotiv of, among others, the *degenerate* shell formulation (cf. [2]) and *multiparameter* shell formulations (cf. [21]). This also established the strain-driven algorithms as standard (e.g. [39,41,24]) since isoparametric finite elements are displacement-based, strain is directly available. Some important contributions, described in the books by Belytschko, Liu and Moran [15] and Bathe [13] mention the decoupling. In this sense, a component perspective on discretization methods was introduced by Areias et al. [11] and a formalization of a general framework based on $\boldsymbol{F}_e\boldsymbol{F}_p$ decomposition was introduced by Areias et al. [7] after a first work focusing on smoothing the complementarity condition of elasto-plasticity [10]. However, that approach requires the inversion of fourth-order tensors, in contrast with the present contribution.

In this work, we propose a simplification of the finite-strain constitutive algorithms with different parameterizations of equilibrium and reference configurations.

Some considerations are required to contextualize the present work:

1. In many element formulations, stress or strain conditions require the use of a local frame, such as beam and shell elements. Dimensional reduction (either strain, such as in the plane-strain case, or stress, such as plane-stress and shell cases) requires specific treatment in the finite strain case. An in-depth study concerning the incompressibility constraint was

* Corresponding author at: Department of Physics, University of Évora, Colégio Luís António Verney, Rua Romão Ramalho, 59, 7002-554 Évora, Portugal.
E-mail address: pmaa@uevora.pt (P. Areias).

http://dx.doi.org/10.1016/j.engstruct.2016.06.035
0141-0296/

performed by Antman and Schuricht [3]. For the discrete shell case with the correct thickness extensibility, Hughes and Liu [27] were the first to recognize the need for a specific treatment. We here show a general methodology to treat known stress or strain components in finite strain problems.

2. Large amplitude motions with elements containing rotational degrees-of-freedom can take computational advantage of a equilibrium formulation based on relative strains, circumventing the use of *total* rotation degrees-of-freedom. Alternatives are more unmanageable, specially in the constrained case (e.g. [18]). It is known that some commercial codes store quaternion parameters to avoid the singularities in large amplitude rotations.
   Our consistent updated-Lagrangian method (cf. [6,12,9]) is extended to avoid the storage of rotation matrices.
3. Due to their complementarity form (see, e.g. [31]), plasticity problems often exhibit convergence difficulties for large strain values and severe sensitivity to step size [7].
   We here present two measures to attenuate these difficulties: the use of a smoothed complementarity condition and the removal of iterative rotation matrices from the constitutive laws in finite strains.
4. Anisotropic constitutive laws make use of a constitutive frame which must be related to the aforementioned local frame.
   Even in the absence of dimensional reduction, a local frame is often required for the representation of anisotropic behavior. Among other relevant properties, the proposed algorithm remains valid for anisotropic hyperelasticity,
5. It is computationally convenient that both hyperelastic and finite-strain elasto-plastic laws are implemented in a unique algorithm and applicable to any discretization scheme.
   This technique is introduced here by specializing the frame-of-reference.
6. Classical assumed-strain elements typically do not directly provide the deformation gradient (e.g. [16]), necessary for many constitutive formulations.
   An estimated deformation gradient can be calculated from the polar decomposition if an approximate rotation matrix is available. Since a constitutive frame is adopted, the rotation matrix is obtained from this frame in two distinct configurations.
7. Since the seminal contributions of Weber and Anand [45] and Simo [39], Kirchhoff stress tensors (i.e. $\boldsymbol{\tau} = J\boldsymbol{\sigma}$) are frequently employed in the yield functions. Two fundamental textbooks on this approach are de Souza Neto et al. [24] and Simo and Hughes [41]. The use of Kirchhoff stress tensor is a computational convenience, as commonly adopted elasto-plastic and hyperelastic models are often quasi-incompressible. This is not the case of porous plasticity or metal *elasticity* (cf. [9]).
8. Although theoretically identified, see Shutov and Kreißig [38] by means of a thermodynamically-consistent function (ψ_{kin} in [38]), back-stresses are often introduced in a ad-hoc form. Typically, this requires a frame-invariant integration very similar to the hypoelastic formulations. Some experiments were performed by Areias and Rabczuk (cf. [10]).
9. Semi-implicit formulations, where certain quantities are fixed in the flow vector (but not the flow vector itself), cf. [32,15] can lead to substantial savings in constitutive integration. We further extend the semi-implicit algorithm presented in [7] to achieve very large time steps. Consistent linearization of integrated form of objective rates is intricate and computationally expensive, making it a possible candidate for the *explicit* integration part of the semi-implicit scheme.
   In this work, rigid body motions are *exactly* represented.

A more inclusive approach to constitutive modeling in finite strains, compatible with a variety of finite element discretizations, is henceforth delineated and tested. In summary, Section 2 discusses the constitutive integration algorithm in detail including a test of the adequacy of objective rates in the semi-implicit context, Section 3 presents shell, 2D and 3D examples with both isotropic and anisotropic materials and finally some conclusions are drawn in Section 4.

2. Constitutive integration in finite strains

2.1. Objective rates

In the context of hypoelastic-based elasto-plasticity, objective time-derivatives of spatial stress measures (either Cauchy, $\boldsymbol{\sigma}$, or Kirchhoff, $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ with $J = \det \boldsymbol{F}$ where $\boldsymbol{F}$ is the deformation gradient) are adopted. A comprehensive description of this approach is performed in Chapters 7 and 8 of Simo and Hughes [41]. The goal is to employ a rate version of Hooke's law for metal plasticity. Objections to such model are known (e.g. [39]) but have lost some strength with the seminal work of Lehmann, cf. [30] who proved the equivalence between a specific corotational time-derivative of the Hencky strain (with the *logarithmic* spin) and the strain rate $\boldsymbol{D}$. That work has been extended to establish the equivalence between hypoelasticity with the logarithmic rate and hyperelasticity, cf. [47]. The long standing problem of integrability in hypoelasticity is now solved with the logarithmic rate [49]. However, if the rate version of Hooke's law is maintained, the strong ellipticity condition limits the maximum elastic stretch to the range $[0.21162, 1.39561]$ [20], which is only slightly better than with the Jaumann *spin*. Of course any elastic law could be adopted, but Hooke's law in rate form is computationally attractive. A review paper discussing the use of logarithmic spin in finite strain elasto-plasticity discusses many of these points, cf. [48], see also [4].

Two classical forms of reasoning about frame-invariance and classical stress tensors (i.e. Cauchy and Kirchhoff) are based on (i) transport equation (equivalent to the use of Lie derivative) and (ii) rotation by exponential integration of a pre-established spin. Using classical notation (e.g. [26,41]), and focusing on the Kirchhoff stress tensor, these correspond to either:

- Pull-back the Kirchhoff stress, calculate the time-derivative of the result, and push-forward this derivative (this corresponds to the Lie derivative of the Kirchhoff stress or Truesdell rate).
- Rotationally neutralize the Kirchhoff stress (i.e. rotate-back to a fixed reference configuration) so that rigid-body terms are explicatively absent from the time-derivative. Of course, this is a particular case of the pull-back, replacing the deformation gradient with the rotation.

For the pull-back, we use the Kirchhoff stress $\boldsymbol{\tau}$, the second Piola–Kirchhoff stress $\boldsymbol{S}$ and the deformation gradient $\boldsymbol{F}$:

$$\begin{aligned} \boldsymbol{\tau} &= \boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T \iff \\ \dot{\boldsymbol{\tau}} &= \boldsymbol{L}\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{L}^T + \underbrace{\boldsymbol{F}\dot{\boldsymbol{S}}\boldsymbol{F}^T}_{\check{\boldsymbol{\tau}}} \end{aligned} \tag{1}$$

where $\dot{\boldsymbol{\tau}}$ is the time-derivative of the Kirchhoff stress and $\check{\boldsymbol{\tau}} = \boldsymbol{F}\dot{\boldsymbol{S}}\boldsymbol{F}^T$ is identified as the constitutive or *objective* time-derivative (cf. [43]). In (1), $\boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}$ is the velocity gradient, with its symmetric part being identified as strain rate, $\boldsymbol{D} = 1/2\left(\boldsymbol{L} + \boldsymbol{L}^T\right)$ and its skew-symmetric part being given by the vorticity tensor $\boldsymbol{W} = 1/2\left(\boldsymbol{L} - \boldsymbol{L}^T\right)$. For the rotationally neutralized case, rotations (here identified by the tensor $\boldsymbol{R}$) replace the deformation gradient in (1):

$$\begin{aligned} \boldsymbol{\tau} &\cong \boldsymbol{R}\boldsymbol{S}\boldsymbol{R}^T \iff \\ \dot{\boldsymbol{\tau}} &\cong \boldsymbol{\Omega}\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{\Omega}^T + \underbrace{\boldsymbol{R}\dot{\boldsymbol{S}}\boldsymbol{R}^T}_{\check{\boldsymbol{\tau}}} \end{aligned} \tag{2}$$