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Wrinkling of stretched thin sheets: Is restrained Poisson's effect the sole cause?

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ABSTRACT

It is widely accepted that wrinkling of thin sheets under tension is due to the compressive stresses that emerge in central zone, orthogonally to the direction of applied tension, being caused by the variation of Poisson's effect from the fixed supports to central zone of the sheet. By means of an analytical approach consisting of displacement fields (designated as "modes") that kinematically enrich the solution of pre-wrinkling stress field, this paper shows that Poisson's effect is not the sole cause of wrinkling. Starting from an extensional mode, other modes with physical meaning (associated to Poisson's effect and warping shear) are consecutively added to the final displacement field. This modal approach unveils that transversal compressive stresses, which trigger the sheet wrinkling, are due not only to restrained Poisson's effect (known factor) but also due to warping shear deformation. This identification is the main and original contribution of the present work. The reduction of warping shear in stretched sheets (e.g., via the addition of transversal fibers) could possibly avoid wrinkling phenomena of elastic thin sheets and membranes used in aerospace applications, such as inflatable antennas and solar sails. Additionally, the paper presents approximate analytical solutions of pre-wrinkling fields (displacements, strains, stresses) that are deemed useful to derive future fully analytical formulae for the prediction of critical wrinkling loads.

1. Introduction

Ultra-thin structures have been used for a wide variety of applications in aerospace applications, such as inflatable antennas and solar sails. Ultra-thin structures can be gathered in two main groups: (i) membranes, which have negligible bending stiffness, and (ii) thin sheets, which have non-negligible bending stiffness. The highly reduced thickness of sheets and membranes makes them ideally suitable when high strength vs. low weight demand becomes a crucial parameter. However, their reduced thickness also makes them prone and sensitive to instabilities. Instability phenomenon is a key problem to control when designing ultra-thin structures, because it can reduce the structural performance and affect their safety. It is widely accepted that instability phenomenon takes place when compressive forces are applied. Despite being less known, instabilities may also occur under tensile loading. This is the case of membranes and thin sheets under uniaxial tension. When tension is increasingly applied in one direction of a membrane (or sheet), it suddenly wrinkles in the orthogonal direction and short-length waves emerge [1,2] (see Fig. 1).

During the last decade, this problem has attracted the attention of several researchers worldwide. Wong and Pellegrino [4–6] performed an extensive research on the wrinkling behavior of membranes, which included experimental tests, analytical formulations and numerical simulations. On the basis of the work by Wong and Pellegrino [4–6], Wang et al. [7] proposed a new concept of shell-membrane and its post-wrinkling analysis of clamped rectangular shell-membrane subjected to uniform transverse in-plane displacement. Results from numerical simulation showed good agreement with their analytical predictions. Massabo and Gambarotta [8] formulated the problem of the wrinkling of plane isotropic membranes characterized by a Fung type constitutive model in biaxial tension and solved within the framework of finite strain hyperelasticity.

Another interesting problem is the wrinkling of thin sheets under tensile loading. In case of sheets, the bending stiffness is not negligible and should be considered in the analysis. Friedl et al. [9] discussed the phenomenon of sheet buckling under tension for the case in which the instability is a result of the clamped boundary conditions, preventing transversal displacements along the loaded edges. They proposed approximate analytical formulas to estimate the tensile loading that causes instability as well as the minimum half wave numbers in transversal direction. Their

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Fig. 1. Stretch-induced wrinkling in sheet – Nayyar et al. [3], permission by Elsevier.

analytical expressions are based on the assumptions of (i) uniform pre-buckling stress field and (ii) single parameter correlation between longitudinal tensile stresses and transversal compressive stresses. Cerda and Mahadevan [10] and Cerda et al. [11] conducted some experiments on polyethylene sheets with clamped lateral edges and different lengths and developed a simple formula that relates the evolution of post-buckling axial strain (extension) with the half-wavelength of wrinkles. Their analytical estimates matched the experimental results, but no formula was given for the critical strain (the latter was given by Friedl et al. [9] in the context mentioned above). Jacques and Potier-Ferry [12] also developed analytical expressions to evaluate the critical tensile load of clamped sheets. However, they assumed that (i) the longitudinal tensile stresses are uniform and (ii) the transversal compressive stresses do not vary in the transversal direction but vary in the longitudinal direction according to a single function. They determined an analytical solution for uniform transversal compressive stresses but claimed the problem has no analytical solution if these stresses vary along the longitudinal x-axis. In order to determine the exact shape of stress function, as well as the ratio between the transversal compressive stresses and applied tensile load, Jacques and Potier-Ferry [12] performed linear finite element simulations. Their analytical predictions (using numerically determined functions) were in good agreement with finite element results, both for critical tensile force and mode shape. Recently, Nayyar et al. [3] presented an excellent numerical study on stretch-induced wrinkling of hyperelastic thin sheets based on nonlinear finite element analyses. They determined stretch-induced stress distribution patterns in the elastic sheets, assuming no wrinkles. They found that appearance of compressive stresses in the transverse direction, which is a prerequisite for wrinkling, depended on both the length-to-width aspect ratio of the sheet and the applied tensile strain in the longitudinal direction. Depending on these parameters, Nayyar et al. [3] proposed a phase diagram that comprises four different distribution patterns of the stretch-induced compressive stresses. Even more recently, Nayyar et al. [13] measured experimentally stretch-induced wrinkle patterns in thin polyethylene sheets using the three-dimensional digital image correlation technique. They observed the wrinkle amplitude first increases and then decreases with increasing longitudinal strain, in agreement with results of finite element simulations of a hyperelastic thin sheet.

Despite these high quality works, little has been done to understand the mechanical roots (onset) of wrinkling phenomena, since it is believed that restrained Poisson's effect is the sole cause of it. Furthermore, analytical expressions to characterize the non-uniform stress state at the onset of wrinkling are yet to be presented. Additionally, a formula to calculate the critical load (or strain) using a true (real) non-uniform stress state is not yet available. Existing formulas for the calculation of critical load [9,12] are either (i) established on too unrealistic assumptions (uniform stress distributions) or (ii) based on the calibration of parameters with limited range (sheet slenderness and aspect ratio). All available analytical approaches to estimate the wrinkling strain assume an *uniform* pre-wrinkling state [14,15]. Indeed, a uniform pre-wrinkling stress/strain field is often assumed for the entire sheet, arising from (i) displacements $u = \varepsilon \cdot x$ in the longitudinal sheet direction (proportional to the imposed strain ε) and (ii) displacements $v = -\nu \cdot \varepsilon \cdot y$ in the transversal sheet direction (proportional to ε and Poisson's ratio ν). Obviously, this pre-wrinkling uniform state can only be assumed "*away from the edges*" of the sheet. Owing to the complexity and non-uniformity of pre-buckling fields, most of the research works base their findings through fully numerical results (finite element simulations), which are rigorous but lack a mechanically-based reasoning.

The first objective of this paper is to understand the origins of transversal compressive stresses that trigger wrinkling of stretched sheets. It is widely accepted that the emergence of transversal compressive stresses is a consequence of restraining Poisson's effect due to sheet clamped edges. However, from the kinematical viewpoint, still there is not a clear path of coupling between the imposed longitudinal strain due to tensile loading and the remaining field components. The modal formulation presented in this paper intends to shed light on this issue. The second objective of this paper is to obtain analytical solutions of the pre-buckling non-uniform fields (displacements, strains, stresses) that are deemed necessary to determine the critical wrinkling strain analytically. Hopefully, these analytical solutions of pre-buckling stress fields will be helpful to derive formulas capable of rigorously predicting the wrinkling critical strain of stretched sheets.

2. Formulation

The model is a rectangular sheet under uniaxial stretching, i.e., tensile loading along its longitudinal x-axis. Let us consider the coordinate system (x, y, z) and the displacement components u, v and w, written in this coordinate system, as indicated in Fig. 1(a). The sheet depicted in Fig. 1(a) has two longitudinal edges located at $x = 0$ and $x = a$ (sheet length is a) and two lateral edges located at $y = \pm b/2$ (sheet width is b), which are fully free. The edge $x = 0$ is fully clamped, as no in-plane displacements are allowed ($u(0) = v(0) = 0$). The edge $x = a$ is fully rigid ($v(a) = 0$) and only uniform axial displacement is allowed ($du/dy = 0$ at $x = a$).

The linear strain–displacement (kinematical) relations are well known,

$$\begin{aligned} \varepsilon_{xx} &= u_{,x} \\ \varepsilon_{yy} &= v_{,y} \\ \gamma_{xy} &= u_{,y} + v_{,x}, \end{aligned} \tag{1}$$

where $(\cdot)_{,x} \equiv d(\cdot)/dx$ and $(\cdot)_{,y} \equiv d(\cdot)/dy$.

Let the displacement field be represented by

$$\begin{aligned} u &= \varphi_i u_i \\ v &= \varphi_i v_i, \end{aligned} \tag{2}$$

where (i) the summation convention applies to subscript i ($i = 1, 2, 3, \ldots$), (ii) $u_i(y)$, $v_i(y)$ and $w_i(y)$ stand for the displacement functions varying transversally along y-axis (sheet width b – see Fig. 2), (iii) $\varphi_i(x)$ are amplitude functions defining the longitudinal variation of $u_i(y)$, $v_i(y)$ and $w_i(y)$ along x-axis (sheet length a – see Fig. 2). It should be highlighted that each set of these functions $(u_i(y), v_i(y), w_i(y))$ is designated hereafter as "*mode i*" ($i = 1, 2, 3, \ldots$). Using (1) and (2), it is a straightforward mater to obtain the strain components as

$$\begin{aligned} \varepsilon_{xx} &= u_i \varphi_{i,x} \\ \varepsilon_{yy} &= v_{i,y} \varphi_i \\ \gamma_{xy} &= u_{i,y} \varphi_i + v_i \varphi_{i,x}, \end{aligned} \tag{3}$$