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Effects of approximations on non-linear in-plane elastic buckling and postbuckling analyses of shallow parabolic arches

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ABSTRACT

In the non-linear in-plane elastic buckling and postbuckling analyses of shallow parabolic arches, to overcome the difficulty in deriving an accurate expression for the non-linear normal strain, an approximation assumption that the derivative of the vertical coordinate with respect to the horizontal coordinate satisfies $(\mathrm{d}y/\mathrm{d}z)^2 \ll 1$ has been adopted in many investigations. The merit of the assumption is that it leads to the same differential equations of equilibrium and the same solutions as those for shallow circular arches. However, the accuracy of the assumption and the limitation of the analytical solutions have not been examined and because of the approximation, the analytical solutions may lead to significant errors for the buckling loads of shallow parabolic arches in some cases. This paper investigates the effects of the approximation assumption on the accuracy of in-plane buckling and postbuckling analyses of pin-ended and fixed shallow parabolic arches by comparing the analytical solutions with their finite element counterparts. It is found that the analytical solutions based on the assumption have some limitations because the assumption holds approximately only for extremely shallow parabolic arches, but is not valid for most shallow parabolic arches. The analytical solutions for the buckling loads based on the assumption are larger than the corresponding finite element results for parabolic arches with a rise-to-span ratio greater than 0.08, the error of the analytical solution increases with an increase of the rise-to-span ratio of the arch, and the sources for the errors are identified and discussed. Hence, caution should be exercised when using the analytical solutions to predict the buckling load of shallow parabolic arches, particularly of those with a rise-to-span ratio greater than 0.08.

1. Introduction

A number of studies of in-the plane buckling and postbuckling behavior of arches have been reported and most of them focus on the circular arches [1–14]. The analytical solutions for the in-plane buckling of high circular arches were obtained by Hodges [1] and verified to be accurate later by Chang and Hodges [2] using a finite element (FE) analysis, while the analytical solutions for the non-linear in-plane buckling and postbuckling of shallow circular arches under a uniform radial load and under a central concentrated load were derived by Pi et al. [3] and Bradford et al. [4] respectively and the solutions were also verified by the FE analyses.

In addition to circular arches, the parabolic arches are also widely used in engineering structures, particularly in bridge construction. The in-plane buckling and postbuckling behavior of parabolic arches have been investigated by several researchers numerically, analytically and experimentally [15–18]. The profile of the parabolic arch can be described in the rectangular axis system oyz as [17,18] (Fig. 1)

$$y = \frac{1}{2p}\left[z^2 - \left(\frac{L}{2}\right)^2\right], \quad z \in \left[-\frac{L}{2}, \frac{L}{2}\right] \tag{1}$$

where L is the span of the arch, the origin o of the axis system is located at the centre between the two ends of the arch, with the positive direction of the axis oy being vertically downward and the positive direction of the axis oz being toward the right hand end of the arch, and p is the focal parameter of the parabolic arch defined by

$$p = \frac{L^2}{8f}, \tag{2}$$

with f being the rise of the arch.

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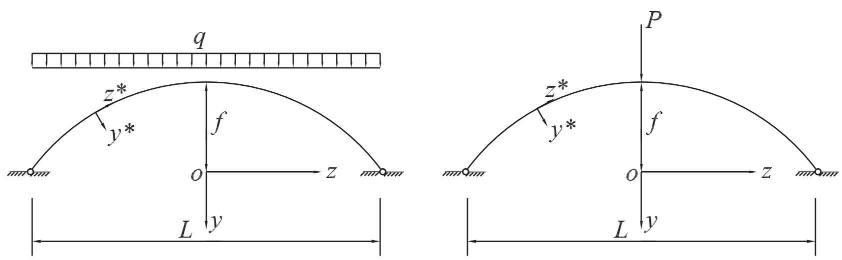

Fig. 1. Parabolic arches.

In deference to the case of circular arches [1–4], it is difficult to formulate the accurate non-linear longitudinal normal strain for parabolic arches and consequently it is also difficult to derive the accurate analytical solutions for the non-linear in-plane buckling of parabolic arches. Hence, approximate analytical solutions were sought by several researchers [17,18,31–34] using an approximation assumption that

$$\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2 \ll 1 \tag{3}$$

in derivation of the non-linear longitudinal normal strain for shallow parabolic arches, which have been used in large span roof structures as reported in [35] and bridges as reported in [36].

This assumption of Eq. (3) leads to an equivalent assumption as

$$\mathrm{d}s = \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2}\,\mathrm{d}z \approx \mathrm{d}z, \tag{4}$$

which is also used in the derivation of the non-linear strain of shallow parabolic arches [17,18,31–34].

The merit of these assumptions is that they lead to the same differential equations of equilibrium and the same analytical solutions for shallow parabolic arches as those for shallow circular arches. Because analytical solutions for the non-linear in-plane elastic buckling and postbuckling of shallow circular arches with different boundary conditions under different loads are available in the open literature [3,4,19–30], these approximation assumptions make analytical investigations of the non-linear buckling and postbuckling of shallow parabolic arches quite straightforward. Although the assumptions given by Eqs. (3) and (4) have these merits, they may have disadvantages and limitations. From Eq. (1), it follows that

$$\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2 = \frac{z^2}{p^2}, \tag{5}$$

from which the assumptions given by Eqs. (3) and (4) hold only approximately near the crown of the arch (with a very small coordinate z) and for extremely shallow parabolic arches (with a large focal parameter p), but do not hold for most shallow parabolic arches. Hence, the non-linear strain derived based on the assumptions given by Eqs. (3) and (4) may have significant errors when an arch is not extremely shallow. Consequently, analytical solutions based on the assumptions may erroneously predict the in-plane elastic buckling load and postbuckling response of most shallow parabolic arches used in engineering structures. However, the disadvantages and limitations of analytical solutions based on the approximation assumptions given by Eqs. (3) and (4) have not been addressed adequately, and the analytical solutions have been used incorrectly for parabolic arches that are not very shallow [31–34]. Hence, identifying the effects of the approximation assumptions on the accuracy of the solutions is much needed, as well as determining the limits of the rise-to-span ratio within which the analytical solutions can be used for predicting the non-linear in-plane buckling and postbuckling of parabolic arches closely, so that incorrect use of the analytical solutions can be avoided.

This paper investigates the effects of the approximation assumptions on the accuracy of analytical solutions for the in-plane buckling and postbuckling responses of parabolic arches under a uniform vertical load or a central concentrated load (Fig. 1) by comparing the analytical solutions with FE results. The merits and disadvantages of the assumptions are examined, the limitations of the analytical solutions are clarified, and the maximum value of the rise-to-span ratio of parabolic arches is determined for the analytical solutions to be able to predict the non-linear in-plane buckling and postbuckling response. The sources for the errors in the analytical solutions produced by the approximation assumptions are also identified.

2. Non-linear analysis under a uniform vertical load

When a shallow parabolic arch is subjected to a uniform vertical load q over its full span (Fig.1a), under the assumptions given by Eqs. (3) and (4), the solution for the vertical displacement v can be obtained as [17] (Appendix A)

$$v = \frac{Q}{\mu_1^2 p}\left\{\frac{\cos(\mu_1 z)}{\cos(\mu_1 L/2)} - 1 + \frac{1}{2}\left[(\mu_1 z)^2 - (\mu_1 L/2)^2\right]\right\} \tag{6}$$

for pin-ended arches, and

$$v = \frac{Q}{\mu_1^2 p}\left\{\frac{\mu_1 L/2\left[\cos(\mu_1 z) - \cos(\mu_1 L/2)\right]}{\sin(\mu_1 L/2)} - 1 + \frac{1}{2}\left[(\mu_1 z)^2 - (\mu_1 L/2)^2\right]\right\} \tag{7}$$

for fixed arches, where the horizontal force parameter μ_1 and the dimensionless load Q are defined by

$$\mu_1^2 = \frac{N}{EI_x} \quad \text{and} \quad Q = \frac{qp}{N} - 1, \tag{8}$$

and where E is the Young's modulus of the material, I_x is the second moment of area of the cross-section about its major principal axis, and N is the horizontal force.

By defining $\mu^2 = \mu_1^2 p^2$, $\theta = z/p$, $\Theta = L/2p$, and $\beta = \mu\Theta$, it can be deduced that $\mu = p\mu_1$ and $\beta = \mu\Theta = \mu_1 L/2$ and the vertical displacement v given by Eqs. (6) and (7) can then be rewritten as

$$v = \frac{Qp}{\mu^2}\left\{\frac{\cos(\mu\theta)}{\cos\beta} - 1 + \frac{1}{2}\left[(\mu\theta)^2 - \beta^2\right]\right\} \tag{9}$$

for pin-ended arches, and

$$v = \frac{Qp}{\mu^2}\left\{\frac{\beta[\cos(\mu\theta) - \cos\beta]}{\sin\beta} - 1 + \frac{1}{2}\left[(\mu\theta)^2 - \beta^2\right]\right\} \tag{10}$$

for fixed arches.

It is noted that when the focal parameter p of a parabolic arch is replaced by the radius R of a circular arch, and the horizontal coordinate z and span L of the parabolic arch are replaced by the axial curvilinear coordinate s and length S of the circular arch, the