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A finite strain quadrilateral based on least-squares assumed strains

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ABSTRACT

When compared with advanced triangle formulations (e.g. Allman triangle and Arnold MINI), specially formulated low order quadrilateral elements still present performance advantages for bending-dominated and quasi-incompressible problems. However, simultaneous mesh distortion insensitivity and satisfaction of the Patch test is difficult. In addition, many enhanced-assumed (EAS) formulations show hourglass patterns in finite strains for large values of compression or tension; EAS elements often present convergence difficulties in Newton iteration, particularly in the presence of high bulk modulus or nearly-incompressible plasticity. Alternatively, we discuss the adequacy of a new *assumed-strain* 4-node quadrilateral for problems where high strain gradients are present. Specifically, we use relative strain projections to obtain three versions of a selectively-reduced integrated formulation complying a priori with the patch test. Assumed bending behavior is directly introduced in the higher-order strain term. Elements make use of least-square fitting and are generalization of classical $\overline{\boldsymbol{B}}$ and $\overline{\boldsymbol{F}}$ techniques. We avoid ANS (assumed natural strains) by defining the higher-order strain in *contravariant/contravariant* coordinates with a fixed frame. The kinematical part of the constitutive updating is based on *quadratic* incremental Green–Lagrange strains. Linear tests and both hyperelastic and elasto-plastic constitutive laws are used to test the element in realistic cases.

1. Introduction

Low order elements are still the preferred choice for non-linear and possibly non-smooth simulations, such as the ones including contact, friction and fracture. We are specially concerned with these themes and have been using a variant of the MINI triangle [9] to perform ductile fracture simulations where the equations correspond to near-incompressibility and are often non-smooth (cf. [8,4,6]). However, it is widely established that specially formulated quadrilaterals often present higher resolution for localization problems. Enhanced assumed strain (EAS) elements are typically presented as a solution (cf. [41,2]) but for large compressive or tensile strains, hourglass instabilities can occur (cf. [42,24,21]). The issue of stabilization of EAS elements has been discussed during the late 20th century and early 21st century, cf. [2]. Mesh distortion sensitivity is also a widely discussed concern with EAS and has been remedied by the first Author in [2]. The cost-effectiveness of EAS elements for large-scale problems is another aspect that deserves a careful attention (remarkably, the work of Puso [37] addresses this problem). Finally, a seldom addressed problem, implicitly addressed in the seminal work of Simo, Armero and Taylor (cf. [42]) is the difficult Newton iteration convergence often exhibited by EAS elements. These issues have stirred interest in different approaches, such as Cosserat-point formulations, which again still show some mesh distortion sensitivity, cf. [27]. After a long period of experimentation we concluded that mesh distortion sensitivity for specially formulated quadrilaterals could be improved by selecting the appropriate local frames. It is worth mentioning that there are recent elements that do not comply with the Patch-test (e.g. area or volume coordinate elements, cf. [26]) and produce exact results in very specific circumstances but fail in other cases [36].

Requirements for our low order quadrilateral formulation are:

- Straightforward use with finite strain plasticity and localization problems;
- Objectivity (both nodal permutation and finite strain);
- Satisfaction of the Patch-test;
- Competitive coarse-mesh accuracy for linear bending problems;

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- Absence of locking with nearly incompressible problems;
- Absence of hourglassing in severe compression and tension tests;
- Absence of internal degrees-of-freedom and static condensation.

Despite the relative success of EAS and Cosserat-point approaches, we here take a different approach: starting with a variant of the $\overline{\boldsymbol{B}}$ element (for finite strains, see Simo et al. [46]) and an exercise proposed in T. Hughes' book (cf. [22], page 261) we generalize the idea to create a new finite strain element. It is worth noting that a preliminary attempt was made by Zhu and Cescotto [48] for general assumed strain element, but without least-square fitting of strains. For selectively-reduced integration, a number of shortcomings emerged for distorted elements when a single central frame was attempted, a fact that led us to intensively explore variants of the original idea by Hughes. The use of least-squares (or equivalently either the Hellinger–Reissner variational principle, see page 285 of [51] or, with a Lagrange multiplier field, the Hu–Washizu variational principle, see [43]) allows, after a the introduction of assumed-strains in contravariant/contravariant coordinates, the generalization of this original idea. Bending behavior is directly introduced in contravariant/contravariant coordinates and fitted with a weighed least-square approach. The weight matrix is a function of the Poisson coefficient. Several linear benchmarks and nonlinear tests are performed, comparing the results of the proposed methodology with well established formulations that are known to comply with the Patch test. This work is divided as follows: Section 2 presents the equilibrium equations for arbitrary configurations, element formulations and corresponding eigenvalue tests and the constitutive model. Section 3 shows a comparative study of linear versions of the elements, where excellent results can be observed. Section 4 presents the nonlinear tests and finally, in Section 5, conclusions are drawn concerning the best performing element variant.

2. Governing equations

2.1. Equilibrium for an arbitrary reference configuration

Cauchy equations of equilibrium for an arbitrary reference configuration are obtained by manipulation of the spatial version of equilibrium (the derivations for the latter are shown in Ogden [31] and extended here). We write the spatial version of Cauchy equations as:

$$\nabla \cdot \boldsymbol{\sigma}^T + \boldsymbol{b} = \boldsymbol{0} \tag{1}$$

with $\boldsymbol{\sigma}$ being the Cauchy stress in an orthonormal basis and $\boldsymbol{b}$ the body force vector in the same basis. Eq. (1) is satisfied for a pseudo-time parameter $t_a \in [0, T]$ with T being the total load parameter of the analysis and for a position $\boldsymbol{x}_a \in \Omega_a$ belonging to the deformed position domain (here identified as Ω_a). Complementing (1), essential and natural boundary conditions defined in terms of two functions g_i and h_i are required (cf. [22]):

$$u_i = g_i \quad \text{on} \quad \Gamma_{g_i} \tag{2}$$

$$\sigma_{ij} n_j = h_i \quad \text{on} \quad \Gamma_{h_i} \tag{3}$$

with the boundary $\Gamma_a = \partial\Omega_a$ being partitioned as $\Gamma_a = \Gamma_{\boldsymbol{g}} \cup \Gamma_{\boldsymbol{h}}$ (essential and the natural parts of the boundary, respectively). In Eq. (3), n_j are components of the outer normal to Γ_a. Identifying the deformation gradient as $\boldsymbol{F}$, it is possible to use derivatives with respect to undeformed coordinates; a manipulation of (1) with the use of the second Piola–Kirchhoff stress, $\boldsymbol{S}$, allows the writing the alternative material form of equilibrium:

$$\nabla_0 \cdot (\boldsymbol{FS})^T + J\boldsymbol{b} = \boldsymbol{0} \tag{4}$$

or, using ∇_0 as the material gradient operator (the derivative with respect to $\boldsymbol{x}_0$). In (4), $J = \det \boldsymbol{F}$. We now consider two consecutive configurations Ω_a and Ω_b at instants t_a and t_b with $t_a \geqslant t_b$. A generalization of (4) then follows:

$$\nabla_b \cdot (\boldsymbol{F}_b \boldsymbol{S}_b)^T + J_b \boldsymbol{b} = \boldsymbol{0} \tag{5}$$

where

$$\nabla_b = \frac{\partial}{\partial \boldsymbol{x}_b} \tag{6}$$

$$\boldsymbol{F}_b = \nabla_b \cdot \boldsymbol{x}_a \tag{7}$$

$$J_b = \det \boldsymbol{F}_b \tag{8}$$

and

$$\boldsymbol{S}_b = J_b \boldsymbol{F}_b^{-1} \boldsymbol{\sigma} \boldsymbol{F}_b^{-T} \tag{9}$$

If a pseudo-time instance t_a belongs to the interval $[0, T]$, we can re-write (5) as:

$$\nabla_b \cdot (\boldsymbol{F}_{ab} \boldsymbol{S}_{ab})^T + J_{ab} \boldsymbol{b} = \boldsymbol{0} \tag{10}$$

where $\boldsymbol{F}_{ab} = \nabla_b \boldsymbol{x}_a$ and $\boldsymbol{S}_{ab}^T = \boldsymbol{S}_{ab}$. This conclusion will be used in the weak form of equilibrium. A fact worth pointing out is the following: t_a must be an equilibrium instant, in contrast with t_b. In the context of finite element analyses, this corresponds to quadratic incremental strains.

2.2. Kinematics and stress integration

Eq. (10) and conditions (2) and (3) are adopted. The relative deformation gradient between two configurations Ω_a and Ω_b is given by (note that scalar components of $\boldsymbol{F}_{ab}$ are introduced as $[\boldsymbol{F}_{ab}]_{ij}$ for the ith row and jth column):

$$\boldsymbol{F}_{ab} = \frac{\partial \boldsymbol{x}_a}{\partial \boldsymbol{x}_b} \tag{11}$$

or, using the covariant basis (cf. [7]), the following product is obtained:

$$\boldsymbol{F}_{ab} = \boldsymbol{J}_a \boldsymbol{J}_b^{-1} \tag{12}$$

where $\boldsymbol{J}_a$ is the Jacobian matrix of $\boldsymbol{x}_a$ as a function of curvilinear coordinates ξ:

$$\boldsymbol{J}_a = \frac{\partial \boldsymbol{x}_a}{\partial \xi} \tag{13}$$

In (13), ξ is a set of appropriate curvilinear coordinates. The inverse of the deformation gradient is obtained by swapping indices a and b : $\boldsymbol{F}_{ab}^{-1} = \boldsymbol{F}_{ba}$. The Jacobian determinant, using the same notation, is given by:

$$J_{ab} = \det \boldsymbol{F}_{ab} \tag{14}$$

The spatial covariant metric is defined as:

$$\boldsymbol{m}_{aa} = \boldsymbol{J}_a^T \boldsymbol{J}_a \tag{15}$$

Using the spatial metric we can write the right Cauchy–Green tensor (see [31] for the nomenclature) between two configurations a and b directly obtained from its definition (12) as:

$$\boldsymbol{C}_{ab} = \boldsymbol{J}_b^{-T} \boldsymbol{m}_{aa} \boldsymbol{J}_b^{-1} \tag{16}$$

from which a relative Green–Lagrange strain tensor is obtained:

$$\boldsymbol{E}_{ab} = \frac{1}{2}(\boldsymbol{C}_{ab} - \boldsymbol{I}) \tag{17}$$