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# Second-order stiffness matrix and load vector of an imperfect beam-column with generalized end conditions on a two-parameter elastic foundation

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ABSTRACT

The stiffness matrix and load vector for an imperfect Euler–Bernoulli beam-column with generalized end conditions subjected to axial and transverse loads are presented. The proposed method includes the effects of initial imperfections (i.e., out-of-straightness, out-of-plumbness, and axial load eccentricities at both ends), a two-parameter elastic foundation, partially restrained sidesway and rotational semirigid connections at both ends, and transverse and end axial loads (tension or compression) on the stiffness matrix and load vector. The proposed method is capable of solving the second-order response and lateral stability, and capturing the phenomenon of deflection reversals in 2D framed structures by using a single segment per element. The effects of shear deformations and torsion along the member are not included in the present research. Three comprehensive examples are provided to show the effectiveness and validity of the proposed matrix method.



## 1. Introduction

The second-order and lateral stability analyses of structures made of beam-columns on elastic foundation are of great importance in structural and geotechnical engineering (e.g. buried pipelines, railroad tracks, and foundation systems for buildings and bridges). However, to include the continuity of the soil, multiple-parameter models have been developed by different authors. Perhaps, the Pasternak model has been the most utilized model during recent years which includes transverse springs (Winkler's model) connected by a shear layer [1].

Beam-columns on elastic foundations under transverse and axial loads have been studied independently by numerous authors. Timoshenko and Gere [2] analyzed the buckling of a bar with hinged connections on an elastic foundation subject to a concentric axial load applied at both ends and a distributed axial load along the member. Hetenyi [1] presented a classic procedure for the elastic stability of hinged and clamped beam-columns supported on elastic foundation developing closed form solutions for particular cases like free-free, hinged-hinged and clamped-clamped. Jones [3] presented a model for beams on elastic foundation (Winkler type) under transverse load with free, hinged, clamped ends and rotational end restraints utilizing the finite difference method. Morfidis and Avramidis [4] introduced the stiffness matrix for both Bernoulli and Timoshenko beams including the effects of a two-parameter elastic foundation represented by a spring and a shear layer. Their approach included shear deformations, semi-rigid connections and rigid offsets showing the simplicity of the method compared to the Finite Elements Method (FEM).

The second-order stiffness matrix for a beam-column on elastic foundation was developed by Areiza-Hurtado et al. [5] including the combined effects of bending and shear deformations as well as the shear component of the axial load. Arboleda-Monsalve et al. [6] presented the second-order dynamic stiffness matrix of a beam-column on a two-parameter elastic foundation including transverse time-dependent load and a static axial load as well as the coupling effects of bending and shear deformations. Their proposed matrix method is capable of solving the static, dynamic and stability analyses of framed structures using a single element per member.

On the other hand, the effects of initial geometric imperfections on the second-order and lateral stability of framed structures made of beam-columns have been studied by many researchers. Initial imperfections reduce the axial load capacity of the individual members and that of the frame as a whole causing additional lateral deflections in frame structures subject to lateral and vertical loads and possible premature lateral instability. Razzaq and Calash

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**Nomenclature**

$A_0$, $A_P$, $B_P$ and $C_P$ Constants for the particular solution of the differential Eq. (6)
$b_o, \ldots, b_m$ coefficients of the Fourier series that represent the applied transverse load
$b^2, R^2, F^2, D^2, \overline{Q}, \bar{x}, \overline{S}_a, \overline{S}_b$ dimensionless parameters
$C_1, C_2, C_3, C_4 =$ unknown constants required in the analysis according to the end conditions (see Eq. (17))
$d_1, \ldots, d_n$ coefficients of the initial imperfection represented as a sinusoidal series
$\Delta_o$ initial out-of-plumbness
$E$ elastic modulus of the material
$e_a$ and $e_b$ eccentricities of axial load P at the ends A and B, respectively
$I$ moment of inertia of the beam-column cross section;
$[K_{st}]$ stiffness matrix for analysis under axial and transverse loads
$k_s$ and $k_G$ two parameters of the elastic foundation [ballast modulus $k_s$, and transverse modulus $k_G$]
$\kappa_a$ and $\kappa_b$ stiffness of the rotational connections at A and B, respectively (force × distance/radian)
$L$ beam-column span
$\{M_{EI}\}$ load vector (i.e., fixed-end forces and moments in member AB due to initial imperfection and external forces)
$M_{st}$ bending moment
$\overline{M}_{st}$ dimensionless static bending moment
$P$ axial load applied at the ends of the shear beam-column (+ compression, − tension)
$q(x)$ applied lateral load
$Q$ applied concentrated lateral load
$S_a$ and $S_b$ stiffness indices of the lateral bracings at ends A and B of the shears beam-column, respectively
$V_{st}$ shear force
$\overline{V}_{st}$ dimensionless bending moment
$x$ coordinate along the centroidal axis of the shear beam-column
$y_0(x)$ initial imperfection of the beam-column (see Fig. 2)
$y_1(x)$ static lateral deflection of the centroidal line of the beam-column (see Fig. 2)
$\bar{y}_1(\bar{x})$ dimensionless Static lateral deflection of the centroidal line of the beam-column
$y_{st}(x)$ total static lateral static deflection of the centroidal line of the beam-column (see Fig. 2)

[7] studied the effects of biaxial partial end restraints for nonsway columns with biaxial crookedness and residual stresses. The biaxial semi-rigid end connections had linear, elastic-plastic, or trilinear moment-rotation characteristics. They concluded that residual stresses were less detrimental to column strength than the initial crookedness; and crooked columns with or without residual stresses their strength increase when the degree of end fixities are increased. Chan and Zhou [8] presented the stability analysis of steel members and structures including the effects of initial imperfections on the tangent and secant stiffness matrices using a single element per member. They concluded that the initial imperfections induce adverse effects on the behavior and capacity of beam-column structures becoming less adverse when the structure is controlled by the $P$-$\Delta$ effect and more notable when is controlled by for $P$-$\delta$ effect. Chan and Gu [9] presented the exact tangent and secant matrices for an initial curved beam-column subjected to end axial force and moments using a single element per member. The proposed element can be used as a benchmark element in the second-order analysis of 2D and 3D frames. Later, Chan, Huang and Fang [10] presented a finite element approach for the large-deflection and inelastic analysis of imperfect steel frames with semirigid bases; three approaches were used to describe the effects of the initial imperfections. Results showed that initial geometric imperfections reduce the critical elastic buckling loads and their effect must not be ignored. Xu and Wang [11] analyzed the effects of out-of-straightness and out-of-plumbness on the effective length factor and column strength. They found that the out-of-straightness produced a greater instability on columns when compared to the out-of-plumbness. They concluded that "Given the same initial value of initial geometric imperfection, the influence of the out-of-straightness on the column effective length factor is almost doubled as that of the out-of-plumbness. This finding is consistent with current practice in which the tolerance for the out-of-straightness and the out-of-plumbness are $L/1000$ and $L/500$, respectively".

Smith-Pardo and Aristizabal-Ochoa [12,13] developed expressions and design aids for transverse and longitudinal deflections of an axially restrained beam-column under transverse and axial static loads. The model included the second-order analysis and the effects of initial imperfections represented by a camber. Aristizabal-Ochoa [14] developed a closed-form expression for beam-columns that undergo axial elongation not only from the applied axial forces but also from the transverse deflections. A general solution was derived for the combined effects of end moments, a uniformly distributed load, series of concentrated loads, sidesway and out-of-straightness. Aristizabal-Ochoa [15] also presented a set of second-order slope-deflection equations for beam-columns including the effects of initial curvature, out-of plumbness, axial load eccentricities and semi-rigid connections. His method is capable of capturing the phenomenon of reversals of deflections as the axial loads are increased. More recently, Aristizabal-Ochoa [16] introduced an analytical analysis and closed-form equations to study the second-order response of 2D multi-column systems composed by imperfect Euler-Bernoulli beam-columns with semi-rigid connections. He concluded that the second-order response is strongly affected by the magnitude and sign of initial geometric imperfections, end fixities and lateral bracing. Initial imperfections in multi-column systems can contribute to reversals in the lateral deflections and they might become unstable at lower axial loads when compared to systems with perfectly straight vertical columns.

The main objective of this paper is to present the second-order stiffness matrix and load vector of an imperfect prismatic beam-column including the combined coupling effects of a uniformly distributed two-parameter elastic foundation, eccentric axial loads (tension or compression) applied at both ends, initial geometric imperfections (i.e., initial curvature and out-of-plumbness), transverse loadings, and generalized elastic conditions at both ends of the member (shear and semi-rigid bending connections). The effects of shear deformations and torsion along the member are not included. The proposed method is an extension of the algorithm presented previously by Aristizabal-Ochoa [14–20] and Areiza-Hurtado et al. [5]. The proposed method is capable of capturing the first and second-order behavior, elastic stability of