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A geometrically exact beam model with non-uniform warping coherently derived from the Saint Venant rod

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ABSTRACT

A new geometrically nonlinear model for homogeneous and isotropic beams with generic section including non-uniform warping due to torsion and shear is derived. Each section is endowed with a corotational frame where statics and kinematics are described using a 3D linear elastic model which extends the Saint-Venant solution to non-uniform warping cases. The algebra of change of observer and a mixed variational principle give the model in terms of generalized parameters. Using a mixed interpolation the model is implemented within a FEM Koiter analysis highly sensitive to the geometrical coherence of the formulation.

1. Introduction

Due to their wide use in engineering practice, linear and nonlinear analyses of beam-like members with thin walled or compact sections still represent attractive topics for researchers, who aim to improve both the accuracy of the continuum models and the efficiency of the FEM solution procedures. Over the last few decades, hundreds of works regarding beam models have been published (see [1,2] for a detailed overview of the most important proposals).

Significant contributions regarding the geometric nonlinear analysis of such structures have focused on the determination of the elastic buckling load [3–5], the initial post-buckling behavior [6] and of the whole equilibrium paths [7–12] of single beams and frames.

The present work deals with the formulation of a new geometrically exact model for homogeneous and isotropic beams subjected to variable warping and undergoing large displacements and rotations but small strains. It will be used for the evaluation of the critical and post-critical behavior of assemblages of beams with generic section. Objectivity is an essential prerequisite for the model. In a material formulation this requires that both stress and strain tensors remain unchanged after a rigid body motion or a change of observer. Objectivity is easily satisfied for a 3D Cauchy body, simply by using the Green–Lagrange tensors, but becomes more complex to verify for beams or shells, described through generalized quantities defined over a one dimensional (the beam axis) or two dimensional (the median plane of the shell) domain.

The great majority of beam models are based on the so–called *geometric exact theories* [13–15,2,8,16]. They are geometrically exact being coherently derived by assumed simplified 3D finite kinematics, but tend to produce over-stiff models or to miss some important aspects of the solution already present in the corresponding parent linear model. This is particularly evident if we compare for a 3D beam the constitutive laws, in terms of stress resultants, obtained on the basis of an exact finite kinematic which considers a rigid section motion only and those evaluated using the Saint Venant (SV) rod theory [17]. In particular the shear/torsional coupling present in the linear solution is not recognized and an "a posteriori" correction of the shear and torsional stiffness coefficients is required to avoid locking. Furthermore, as it will be better discussed in Section 3.4, in 3D finite kinematics, it is not simple to include in a coherent way effects already present in the SV solution, as the dependence of the constitutive laws on the Poisson coefficients, the differences between bending and shear principal axes or more complex behaviors due to variable warpings.

To overcome these difficulties in this work we exploit the *Implicit Corotational Method* (ICM) proposed in [18,19], to recover objective nonlinear structural models by reusing the information derived from the 3D continuum solutions that define a corresponding *parent* linear model. The method extends the corotational idea, initially applied to a whole finite element [20–22], at the continuum level. Similarly to [23,24], the introduction of a corotational frame (observer) for each point on the beam axis allows the motion

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to be split in a stretch contribution followed by a pure rotation, according to the decomposition theorem [25]. Using the small strain hypothesis, linear stresses and the displacement gradient defined in this corotational frame provide accurate approximations for the Biot nonlinear stress and strain. These tensors are then introduced in a mixed variational principle in order to obtain the model in terms of generalized parameters (stress/strain resultants) that are expressed in a fixed global frame using a change of observer algebra, once the corotational rotation is appropriately defined.

The nonlinear model so obtained, in particular the constitutive laws in terms of stress resultants, retains all the details of the 3D parent linear solution while its objectivity is ensured implicitly. This is one of the important differences with respect to other formulations based on kinematical descriptions only [26,8,16,9].

The ICM does not require any ad-hoc assumptions about the structural model at hand, nor depends on any particular parametrization of the rotation tensor, but actually behaves as a black-box tool able to translate known linear models into the corresponding nonlinear ones. Moreover, the direct use of a mixed (stress/displacements) description provides an automatic and implicitly coherent methodology for generating models free of nonlinear locking effects, as described in [27–30], and in a format directly suitable for use in FEM implementations.

Linear formulations for beams are widely available in literature. The initial extensions of the SV theory to the non-uniform torsion of thin-walled profiles due to Vlasov [31], has been refined to account for shear stresses due to warping. Starting from the important contributions of Capurso [32] and Tralli [33] a survey on the more recent developments can be found in [1,34–37] and references therein. However many of these proposals only partially account for the richness of 3D continuum introducing appropriate hypotheses about the statics and the kinematics of the body when formulating the one-dimensional model.

Refined beam formulations potentially capable of accounting for the 3D continuum solution of generic materials exactly are based on FEM analyses to derive the cross-section stiffness matrix. Among others, we recall the works of Borri et al. [38,3,39] and the variational-asymptotic method developed essentially by Hodges and coworkers [1,7] also suitable for use in geometric nonlinear cases. For isotropic and homogeneous materials the linear beam model proposed in [40] is a convenient alternative to these general proposals, because it explicitly uses a 3D elastic solution that exactly describes the standard Saint-Venant (SV) behavior while accounts, in a simplified but effective fashion, for the shear/torsion out-of-plane non-uniform warping effects.

In particular the kinematics consider a rigid section motion and an out-of-plane deformation represented by the three SV warpings corresponding to shears and torsion (see [41]) amplified through three independent descriptors variable along the beam axis. The static, used to define the section compliance matrix, is accurately obtained by adding to the exact SV solution some further *secondary* terms due to the variable warping. The axial secondary stress distribution is suggested by the kinematic, while a *Jourawsky*-like approach is used to evaluate the secondary shear stresses contribution through the equilibrium condition in the axial direction. Both the SV and the secondary warping functions are evaluated numerically through a FEM approach (see for instance [40,41]) allowing the section compliance matrix to account for all the coupling effects arising from the 3D problem. The resulting beam model exactly accounts for the SV solution and is valid for the largest variety of cross section (compact, thin/thick walled ones).

A mixed finite element suitable to interpolate stresses and displacements accurately is also presented. It directly uses the equilibrium equation between bi-shear and bi-moment to improve the accuracy and reduce the number of interpolation parameters. By means of a block elimination of the variables that do not require inter-element continuity the element, at the global level, exposes only 9 kinematical parameters per node and uses a pseudo-compatible format to perform the analysis [27,42]. The equilibrium path is recovered by means of a FEM formulation of the Koiter asymptotic approach [43–48], that has shown to be highly sensitive to the geometrical coherence of the structural model and of its finite element interpolation [18,19]. For this reason it constitutes a suitable framework to assess the validity of the proposal.

A series of test cases regarding single beams and frames have been presented and results are compared with those of existing 1D formulations and furnished by shell analyses using the ABAQUS code. A good agreement with the shell model results can be always appreciated also when standard beam formulations, like that employed in ABAQUS, are not accurate.

2. The implicit corotational method

In this section the ICM will be briefly outlined in a simplified form already particularized for a beam continuum while readers can refer to [18,19] for a more detailed discussion of the method.

2.1. Stress and strain tensors in the corotational frame

Let us consider the beam as a Cauchy body [25] referred to the fixed Cartesian frame with origin $\mathcal{O}$ and basis vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ in $\mathbb{E}^3$. Let the reference configuration also be stress-free. Using a material description, the polar decomposition theorem allows the following representation of the deformation gradient $\boldsymbol{F}[\boldsymbol{X}]$ at a material reference point P identified by its position vector $\boldsymbol{X}$:

$$\boldsymbol{F}[\boldsymbol{X}] = \boldsymbol{R}[\boldsymbol{X}]\,\boldsymbol{U}[\boldsymbol{X}], \tag{1}$$

where $\boldsymbol{U}[\boldsymbol{X}]$ is the symmetric and positive-definite stretching tensor and $\boldsymbol{R}[\boldsymbol{X}]$ an orthogonal rotation tensor describing the rigid motion in the neighborhood of $\boldsymbol{X}$.

The position vector $\boldsymbol{X}$ can be split as

$$\boldsymbol{X} = s\boldsymbol{e}_1 + \boldsymbol{x} \tag{2}$$

s being a one-dimensional abscissa along the line axis or *support* and $\boldsymbol{x} = x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3$ lies on the cross section and defines the *fiber* $\Omega[s]$.

It is possible to evaluate, for each fiber, a constant mean rotation $\boldsymbol{Q}$ so expressing the polar decomposition rotation as

$$\boldsymbol{R}[\boldsymbol{X}] = \boldsymbol{Q}\overline{\boldsymbol{R}}[\boldsymbol{X}], \tag{3}$$

where $\overline{\boldsymbol{R}}[\boldsymbol{X}]$ is another rotation. Using Eq. (3), the deformation gradient introduced in Eq. (1) transforms as

$$\boldsymbol{F}[\boldsymbol{X}] = \boldsymbol{Q}\overline{\boldsymbol{F}}[\boldsymbol{X}], \quad \overline{\boldsymbol{F}}[\boldsymbol{X}] = \overline{\boldsymbol{R}}[\boldsymbol{X}]\boldsymbol{U}[\boldsymbol{X}] \tag{4}$$

allowing the motion in the neighborhood of $\boldsymbol{X}$ to be expressed in a deformation $\overline{\boldsymbol{F}}[\boldsymbol{X}]$ followed by a mean rigid motion of the fiber. Letting $\boldsymbol{u}[\boldsymbol{X}]$ be the total displacement of $\boldsymbol{X}$, the following definitions hold

$$\boldsymbol{F}[\boldsymbol{X}] = \boldsymbol{I} + \nabla\boldsymbol{u}[\boldsymbol{X}], \quad \overline{\boldsymbol{F}}[\boldsymbol{X}] = \boldsymbol{I} + \nabla\bar{\boldsymbol{u}}[\boldsymbol{X}], \tag{5}$$

$\nabla = \partial(\cdot)/\partial\boldsymbol{X}$ being the material gradient. The relation between $\nabla\boldsymbol{u}[\boldsymbol{X}]$ and $\nabla\bar{\boldsymbol{u}}[\boldsymbol{X}]$ is simple obtained from Eqs. (4) and (5). From now on the dependence on $\boldsymbol{X}$ or s will be omitted when it is clear from the context.

The material description requires that the strain depends only on $\boldsymbol{U}$ (objectivity) as it happens, for example, using either the *Green–Lagrange strain tensor* $\boldsymbol{\varepsilon}_g = (\boldsymbol{U}^2 - \boldsymbol{I})/2$ or the *Biot strain tensor* $\boldsymbol{\varepsilon}_b = \boldsymbol{U} - \boldsymbol{I}$. $\boldsymbol{\varepsilon}_g$ is simple to evaluate being quadratic in the displacement gradient, its expression in terms of the symmetric and skew part of $\nabla\bar{\boldsymbol{u}}[\boldsymbol{X}], \overline{\boldsymbol{E}}$ and $\overline{\boldsymbol{W}}$ respectively, is