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A simple and robust elastoplastic constitutive model for concrete

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ABSTRACT

An elasto-plastic model for concrete, based on a recently-proposed yield surface and simple hardening laws, is formulated, implemented, numerically tested and validated against available test results. The yield surface is smooth and particularly suited to represent the behaviour of rock-like materials, such as concrete, mortar, ceramic and rock. A new class of isotropic hardening laws is proposed, which can be given both an incremental and the corresponding finite form. These laws describe a smooth transition from linear elastic to plastic behaviour, incorporating linear and nonlinear hardening, and may approach the perfectly plastic limit in the latter case. The reliability of the model is demonstrated by its capability of correctly describing the results yielded by a number of well documented triaxial tests on concrete subjected to various confinement levels. Thanks to its simplicity, the model turns out to be very robust and well suited to be used in complex design situations, as those involving dynamic loads.

1. Introduction

The mechanical behaviour of concrete is rather complex – even under monotonic and quasi-static loading – because of a number of factors: (i) highly *nonlinear* and (ii) strongly *inelastic* response, (iii) *anisotropic* and (iv) eventually *localized* damage accumulation with (v) *stiffness degradation*, (vi) *contractive* and subsequently *dilatant* volumetric strain, leading to (vii) *progressively severe cracking*. This complexity is the macroscale counterpart of several concurrent and cooperative or antagonist micromechanisms of damage and stiffening, as for instance, pore collapse, microfracture opening and extension or closure, aggregate debonding, and interfacial friction. It can therefore be understood that the constitutive modelling of concrete (but also of similar materials such as rock, soil and ceramic) has been the subject of an impressive research effort, which, broadly speaking, falls within the realm of elastoplasticity,[1] where the term 'plastic' is meant to include the damage as a specific inelastic mechanism. In fact, elastoplasticity is a theoretical framework allowing the possibility of a phenomenological description of all the above-mentioned constitutive features of concrete in terms of: (i) yield function features, (ii) coupling elastic and plastic deformation, (iii) flow-rule nonassociativity, and (iv) hardening sources and rules [3]. However, the usual problem arising from a refined constitutive description in terms of elastoplasticity is the complexity of the resulting model, which may lead to several numerical difficulties related to the possible presence of yield surface corners, discontinuity of hardening, lack of self-adjointness due to nonassociativity and failure of ellipticity of the rate equations due to strain softening. As a consequence, refined models often lack numerical robustness or slow down the numerical integration to a level that the model becomes of awkward, if not impossible, use. A 'minimal' and robust constitutive model, not obsessively accurate but able to capture the essential phenomena related to the progressive damage occurring during monotonic loading of concrete, is a necessity to treat complex load situations.

The essential 'ingredients' of a constitutive model are a convex, smooth yield surface capable of an excellent interpolation of data and a hardening law describing a smooth transition between elasticity and a perfectly plastic behaviour. Exclusion of strain softening, together with flow rule associativity, is the key to preserve ellipticity, and thus well-posedness of the problem. In the present paper, an elastopalstic model is formulated, based on the so-called 'BP yield surface' [3,4,11] which is shown to correctly describe the damage envelope of concrete, and on an infinite class of isotropic hardening rules (given both in incremental form and in the corresponding finite forms), depending on a hardening parameter. Within a certain interval for this parameter hardening is unbounded (with linear hardening obtained as a limiting case), while outside this range a smooth hardening/perfectly-plastic transition is described. The proposed model does not describe certain phenomena which are known to occur in concrete, such as for instance

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[1] The research on elastoplastic modelling of concrete has reached its apex in the seventies and eighties of the past century, when so many models have been proposed that are now hard to even only be summarized (see among others, [2,5,8,15]). Although nowadays other approaches are preferred, like those based on particle mechanics [13], the aim of the present article is to formulate a relatively simple and robust model, something that is difficult to be achieved with advanced models.

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anisotropy of damage, softening, and elastic degradation, but provides a simple and robust tool, which is shown to correctly represent triaxial test results at high confining pressure.

2. Elastoplasticity and the constitutive model

This section provides the elastoplastic constitutive model in terms of incremental equations. The form of the yield surface is given, depending on the stress invariants and a set of material parameters. A new class of hardening laws is formulated in order to describe a smooth transition from linear elastic to plastic behaviour.

2.1. Incremental constitutive equations

The decomposition of the strain into the elastic ($\boldsymbol{\epsilon}_e$) and plastic ($\boldsymbol{\epsilon}_p$) parts as

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_e + \boldsymbol{\epsilon}_p \tag{1}$$

yields the incremental elastic strain in the form

$$\dot{\boldsymbol{\epsilon}}_e = \dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}_p. \tag{2}$$

The ‘accumulated plastic strain’ is defined as follows:

$$\pi_a = \int_0^t |\dot{\boldsymbol{\epsilon}}_p| d\tau, \tag{3}$$

where t is the time-like variable governing the loading increments. Introducing the flow rule

$$\dot{\boldsymbol{\epsilon}}_p = \dot{\lambda}\boldsymbol{P}, \tag{4}$$

where $\boldsymbol{P}$ is the gradient of the plastic potential, we obtain that the rate of the accumulated plastic strain is proportional to the plastic multiplier ($\dot{\lambda} \geqslant 0$) as

$$\dot{\pi}_a = \dot{\lambda}|\boldsymbol{P}|. \tag{5}$$

A substitution of Eqs. (2) and (4) into the incremental elastic constitutive equation relating the increment of stress $\dot{\boldsymbol{\sigma}}$ to the increment of elastic strain $\dot{\boldsymbol{\epsilon}}_e$ through a fourth-order elastic tensor $\mathbb{E}$ as $\dot{\boldsymbol{\sigma}} = \mathbb{E}\dot{\boldsymbol{\epsilon}}_e$, yields

$$\dot{\boldsymbol{\sigma}} = \mathbb{E}\dot{\boldsymbol{\epsilon}} - \dot{\lambda}\mathbb{E}\boldsymbol{P}. \tag{6}$$

Note that, for simplicity, reference is made to isotropic elasticity, so that the elastic tensor $\mathbb{E}$ is defined in terms of elastic modulus E and Poisson’s ratio ν.

During plastic loading, the stress point must satisfy the yield condition $F(\boldsymbol{\sigma}, \mathbf{k}) = 0$ at every time increment, so that the Prager consistency can be written as

$$\dot{F} = \boldsymbol{Q} \cdot \dot{\boldsymbol{\sigma}} + \frac{\partial F}{\partial \boldsymbol{k}} \cdot \dot{\boldsymbol{k}} = 0, \tag{7}$$

where $\boldsymbol{Q} = \partial F/\partial \boldsymbol{\sigma}$ is the yield function gradient and $\boldsymbol{k}$ is the hardening parameters vector.

By defining the hardening modulus $H(\boldsymbol{\sigma}, \boldsymbol{k})$ as

$$\frac{\partial F}{\partial \boldsymbol{k}} \cdot \dot{\boldsymbol{k}} = -\dot{\lambda}H(\boldsymbol{\sigma}, \boldsymbol{k}), \tag{8}$$

Eq. (7) can be rewritten in the form

$$\boldsymbol{Q} \cdot \dot{\boldsymbol{\sigma}} - \dot{\lambda}H = 0 \tag{9}$$

Further, the plastic multiplier can be obtained from Eqs. (6) and (9)

$$\dot{\lambda} = \frac{\boldsymbol{Q} \cdot \mathbb{E}\dot{\boldsymbol{\epsilon}}}{H + \boldsymbol{Q} \cdot \mathbb{E}\boldsymbol{P}}. \tag{10}$$

Finally, a substitution of Eqs. (10) into (6) yields the elasto-plastic constitutive equations in the rate form

$$\dot{\boldsymbol{\sigma}} = \mathbb{E}\dot{\boldsymbol{\epsilon}} - \frac{\boldsymbol{Q} \cdot \mathbb{E}\dot{\boldsymbol{\epsilon}}}{H + \boldsymbol{Q} \cdot \mathbb{E}\boldsymbol{P}}\mathbb{E}\boldsymbol{P}, \tag{11}$$

where, for simplicity, the associative flow rule $\boldsymbol{P} = \boldsymbol{Q}$, will be adopted in the sequel.

2.2. The BP yield surface

The following stress invariants are used in the definition of the BP yield function [4]

$$p = -\frac{\mathrm{tr}\boldsymbol{\sigma}}{3}, \quad q = \sqrt{3J_2}, \quad \theta = \frac{1}{3}\cos^{-1}\left(\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}\right), \tag{12}$$

where $\theta \in [0, \pi/3]$ is the Lode’s angle and

$$J_2 = \frac{1}{2}\mathrm{tr}\,\boldsymbol{S}^2, \quad J_3 = \frac{1}{3}\mathrm{tr}\,\boldsymbol{S}^3, \quad \boldsymbol{S} = \boldsymbol{\sigma} - \frac{\mathrm{tr}\boldsymbol{\sigma}}{3}\boldsymbol{I}, \tag{13}$$

in which $\boldsymbol{S}$ is the deviatoric stress and $\boldsymbol{I}$ is the identity tensor.

The seven-parameters BP yield function F [4] is defined as

$$F(\boldsymbol{\sigma}) = f(p) + \frac{q}{g(\theta)}, \tag{14}$$

in which the pressure-sensitivity is described through the ‘meridian function’

$$f(p) = -Mp_c\sqrt{\left(\frac{p+c}{p_c+c} - \left(\frac{p+c}{p_c+c}\right)^m\right)\left[2(1-\alpha)\frac{p+c}{p_c+c} + \alpha\right]}, \quad \text{if} \quad \frac{p+c}{p_c+c} \in [0,1], \tag{15}$$

and $f(p) = +\infty$, if $(p+c)/(p_c+c) \notin [0,1]$. The Lode-dependence of yielding is described by the ‘deviatoric function’ (proposed by [12] and independently by [4])

$$g(\theta) = \frac{1}{\cos[\beta\frac{\pi}{6} - \frac{1}{3}\arccos(\gamma\cos 3\theta)]}. \tag{16}$$

The seven, non-negative material parameters

$$M > 0, \quad p_c > 0, \quad c \geqslant 0, \quad 0 < \alpha < 2, \quad m > 1, \quad 0 \leqslant \beta \leqslant 2, \quad 0 \leqslant \gamma < 1$$

define the shape of the associated yield surface. In particular, M controls the pressure-sensitivity, p_c and c are the yield strengths under ideal isotropic compression and tension, respectively. Parameters α and m define the distortion of the meridian section, while β and γ model the shape of the deviatoric section.

2.3. An infinite class of hardening laws

In order to simulate the nonlinear hardening of concrete, the following class of hardening rules in incremental form is proposed:

$$\dot{p}_c = \frac{k_1}{(1+\delta\pi_a)^n}\dot{\pi}_a, \tag{17}$$

$$\dot{c} = \Omega\dot{p}_c, \tag{18}$$

where four material parameters have been introduced: $k_1 > 0$, $\delta \geqslant 0$, $n > 0$, and $0 < \Omega < 1$.

A substitution of the flow rule (4) into (17) and (18) yields

$$\dot{p}_c = \dot{\lambda}\frac{k_1}{(1+\delta\pi_a)^n}|\boldsymbol{P}|, \tag{19}$$

$$\dot{c} = \dot{\lambda}\frac{\Omega k_1}{(1+\delta\pi_a)^n}|\boldsymbol{P}|. \tag{20}$$

Eqs. (17) and (18) can be integrated in order to obtain the hardening laws in finite form as follows