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Analytical formulation and solution of arches defined in global coordinates

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ABSTRACT

This work deals with the arch defined in global coordinates. It shows the procedure followed to obtain its formulation from the differential system of a curved beam. This procedure consists mainly in applying in the differential system, besides the habitual assumptions of the Strength of Materials, the simplifications of geometric conditions of the planar curves. The resulting model includes the analysis of arches of variable section under any action force, moment, rotation or displacement in its plane. The successive integration of the equations of the system permits obtaining the solution that represents the structural behavior of the arch under any type of support. Applying the boundary conditions of each problem it is obtained directly the exact analytical solution. Examples of calculus of parabolic arches are given to show the practical viability of the procedure followed. Results presented in graphs and tables are comparable with those obtained in the literature. The method is suitable for educational purposes.

1. Introduction

The mechanical behavior of a curved beam, applying the theories of Euler–Bernuolli or Timoshenko, is expressed usually through the equations of equilibrium, constitutive relationships and compatibility equations [1–4] or through compact equations of energy [5–12]. These two ways to annotate the structural behavior have permitted offering results, analytical and/or numerical, only for certain type of pieces. The planar pieces [13–16] more widely studied are with circular axis-line [17,18], parabolic [19–21] and elliptical [22]. The twisted beam more studied is with circular helix axis-line [23–26]. Also the analysis of a curved beam could be concreted using a system of twelve linear ordinary differential equations [27–29]. This joint statement has permitted finding new resolution procedures and thereby broadening the types of pieces to study. But still analytical procedures are limited depending on the complexity of the shape of the axis-line and section of the piece, the characteristics of the material, the action system and the type of support. Both in the case of approaching the study of the pieces by a single differential system or through separate equations (equilibrium and compatibility), functions are expressed in natural coordinates using the Frenet reference system and the variable independent used is the arc length.

Authors that subscribe this investigation, published a general formulation of curved beam elements expressed in the Frenet frame and with independent variable the arc length [30] or with another variable [31], taken into account shearing deformations, variable section, asymmetrical section, generalized loads and any support condition. Later, this general formulation was expressed under a system of global coordinates and with the arc length as independent variable [32], with the objective of representing and interpreting more efficiently, results of internal forces and displacements. In this later document there were shown the advantages of this new formulation compared with the formerly. The principal advantage is the lower triangular nature of the system annotated, that permits the obtaining of analytical results through successive integrations row by row.

As a novelty, the present paper focuses on solving the problem of the arch, because it is a less common type of piece, but of high interest in the design and in analysis of structures. From the differential model in global coordinates, we obtain the exact analytical solution. To do this, first a particular formulation is provided from the general system, representing the arch of symmetrical cross-section loaded into its plane taken into account the axial and shearing deformation and with whatever independent variable. Sometimes, the length of the arch that defines the axis-line is not the fundamental variable of the design. Is remarkable in the formulation and in the examples of this article, the employment of a generic parameter that can be particularized in function of the geometry of the piece element.

The formulation is particularized and solved analytically in two cases of arches which its axis-line is parabolic without taken into account the shearing deformation. In the first example the section is of hyperbolic variation. Data have been chosen to be compared

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with those obtained in the cited literature. In the second example the section is constant and shows how the procedure permits to parameterize and tabulate the obtaining of results. In both cases graphs and tables are offered, useful to specify dimensions, testing and comparison of results.

2. Curved beam formula defined in Global Coordinates

A curved beam is generated by a plane cross section whose centroid sweeps through all the points of an axis curve. The vector radius $\mathbf{r}(s)$ expresses this curved line, where s length of the arch, is the independent variable. The reference coordinate system used here to represent the intervening known and unknown functions of the problem is the Frenet–Serret frame P_{tnb}. Its unit vectors tangent $\mathbf{t}$, normal $\mathbf{n}$ and binormal $\mathbf{b}$ are:

$$(\mathbf{t},\mathbf{n},\mathbf{b}) = (D\mathbf{r}, D^2\mathbf{r}/|D^2\mathbf{r}|, \mathbf{t}\wedge\mathbf{n}) \tag{1}$$

where D is the derivative with respect to the parameter s.

The Frenet–Serret equations [33] describe the movement of the frame system along the axis line. They are obtained with the vectors tangent, normal and binormal derivates with respect to the arch length. Its matricial expression is:

$$D\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix} = \begin{bmatrix}0 & \chi(s) & 0\\ -\chi(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix} \tag{2}$$

where $\chi(s)$ and $\tau(s)$ are the flexure and torsion curvatures respectively.

Assuming the habitual principles and hypotheses of the strength of materials [3] and considering the stresses associated with the normal cross-section (σ,τ_n,τ_b), the geometric characteristics of the section are: area $A(s)$, shearing coefficients $\alpha_n(s)$, $\alpha_{nb}(s)$, $\alpha_{bn}(s)$, $\alpha_b(s)$ and moments of inertia $I_t(s)$, $I_n(s)$, $I_b(s)$, $I_{nb}(s)$.

$E(s)$ and $G(s)$ are the longitudinal and transversal elastic moduli which give the elastic properties of the material.

Equilibrium and kinematics equations compose a system of twelve linear ordinary differential equations of a curved beam element [30].

It is possible to apply a change of basis in the referenced equations and express the functions in a global coordinate system P_{xyz} with unit vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ instead of unit vectors tangent $\mathbf{t}$, normal $\mathbf{n}$ and binormal $\mathbf{b}$, through the direction cosines:

$$\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix} = \begin{bmatrix}\upsilon_{tx} & \upsilon_{ty} & \upsilon_{tz}\\ \upsilon_{nx} & \upsilon_{ny} & \upsilon_{nz}\\ \upsilon_{bx} & \upsilon_{by} & \upsilon_{bz}\end{bmatrix}\begin{bmatrix}\mathbf{i}\\ \mathbf{j}\\ \mathbf{k}\end{bmatrix} \tag{3}$$

Fig. 1 represents the axis-line of a curved beam and its systems of reference, Frenet–Serret and global, associated.

The differential system that governs the structural behavior of a curved beam in global coordinates is expressed as [32]:

$$\begin{array}{l}
DV_x + q_x = 0\\
DV_y + q_y = 0\\
DV_z + q_z = 0\\
-\upsilon_{tz}V_y + \upsilon_{ty}V_z + DM_x + m_x = 0\\
+\upsilon_{tz}V_x - \upsilon_{tx}V_z + DM_y + m_y = 0\\
-\upsilon_{ty}V_x + \upsilon_{tx}V_y + DM_z + m_z = 0\\
-\gamma_{xx}M_x - \gamma_{yx}M_y - \gamma_{zx}M_z + D\theta_x - \Theta_x = 0\\
-\gamma_{xy}M_x - \gamma_{yy}M_y - \gamma_{zy}M_z + D\theta_y - \Theta_y = 0\\
-\gamma_{xz}M_x - \gamma_{yz}M_y - \gamma_{zz}M_z + D\theta_z - \Theta_z = 0\\
-\varepsilon_{xx}V_x - \varepsilon_{yx}V_y - \varepsilon_{zx}V_z - \upsilon_{tz}\theta_y + \upsilon_{ty}\theta_z + D\delta_x - \Delta_x = 0\\
-\varepsilon_{xy}V_x - \varepsilon_{yy}V_y - \varepsilon_{zy}V_z + \upsilon_{tz}\theta_x - \upsilon_{tx}\theta_z + D\delta_y - \Delta_y = 0\\
-\varepsilon_{xz}V_x - \varepsilon_{yz}V_y - \varepsilon_{zz}V_z - \upsilon_{ty}\theta_x + \upsilon_{tx}\theta_y + D\delta_z - \Delta_z = 0
\end{array} \tag{4}$$

The first six rows of the system (4) represent the equilibrium equations. The functions involved in the equilibrium equations are represented in Fig. 2 and its expressions are:

Internal forces

$$\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k} = \int_A \sigma\ dA\mathbf{t} + \int_A \tau_n\ dA\mathbf{n} + \int_A \tau_b\ dA\mathbf{b} \tag{5}$$

Internal moments

$$\mathbf{M} = M_x\mathbf{i} + M_y\mathbf{j} + M_z\mathbf{k} = \int_A (\tau_b n - \tau_n b) dA\mathbf{t} + \int_A \sigma b\ dA\mathbf{n} - \int_A \sigma n\ dA\mathbf{b} \tag{6}$$

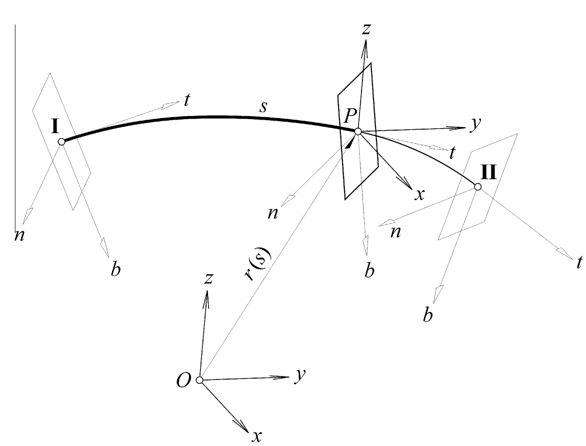

Fig. 1. Curved beam and its reference systems.