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Free vibrations of non-uniform and axially functionally graded beams using Haar wavelets

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ABSTRACT

Vibrations of non-uniform and functionally graded (FG) beams with various boundary conditions and varying cross-sections are investigated using the Euler–Bernoulli theory and Haar matrices. It is assumed that the cross-section and material properties vary along the beam in the axial direction. The system of the governing equations is transformed with the aid of a set of simplest wavelets. To validate the present results, the non-homogeneity of the beams is discussed in detail and the calculated frequencies are compared with those of the existing literature. The results show that the Haar wavelet approach is capable of calculating frequencies for the beams with different shapes, rigidity, mass density, small or large translational and rotational boundary coefficients. The advantage of the novel approach consists in its simplicity, accuracy and swiftness.

1. Introduction

In contemporary engineering conditions, the requirements for structural materials and their properties are becoming more stringent. This is particularly true for the materials which are used in constructional elements or assembly units and are utilized in extremely severe environment or adverse exploitation. Nevertheless, traditional means for improving characteristics and performance of natural materials are depleted. Therefore, an increasing interest in composite materials and the materials with gradients in composition is evident. The tendency is also provoked by economical aspects: extraction and processing of natural resources is limited and expensive.

FG materials withstand high temperatures and resist corrosion. On account of comparatively good fracture toughness, FG materials are less exposed to delamination or cracking in comparison to uniform or homogeneous beams; therefore, FG materials have been under important consideration among engineers in recent decades. A detailed overview of the advanced materials, their development, elemental composition, microstructure, properties, design and application are described in [1] by Byrd. A more comprehensive research on thermoelastic behaviour of FG structures was first conducted by Chakraborty et al. in 2003 yet. Static, free vibrations and wave propagation were investigated by the beam element approach which required an exact solution of the static part of the governing differential equations [2]. Aydogdu and Taskin studied free vibrations of simply supported FG beams with the aid of classical beam theory, parabolic and exponential shear deformation beam theories. The governing equations were found by the Navier type solution [3]. A unified approach for analysing both static and dynamic behaviour of FG beams was proposed by Li extending the Timoshenko beam theory [4]. A fundamental frequency analysis using different higher-order beam theories was carried out by Simsek [5]. In 2010, Alshorbagy et al. suggested FEM for calculating dynamic characteristics of FG beams with material graduation in axially or transversally through the thickness based on the power law [6]. The same method was applied by Shahna et al. for stability analysis of FG tapered Timoshenko beam [7]. Xiang and Yang studied forced vibrations of a three-layer laminated FG Timoshenko beam with arbitrary end supports and varying thickness due to the applied heat [8]. Recently Simsek and Kocatürk studied dynamic behaviour of FG simply-supported beams under a concentrated moving harmonic load. The approach was based on Lagrange's equations [9]. Bending and vibration of cylindrical beams with arbitrary radial non-homogeneity were investigated by Huang and Li [10]. A dynamic system with a moving mass was broadly studied by Simsek in [11,12], and Khalili et al. [13]. A new approach for calculating free vibration of FG beams with non-uniform cross-section area and varying physical properties along its longitude was proposed by Huang and Li [14] last year. The approach was based on the Fredholm integration equation.

The analytical method for studying free vibrations of FG beams was provided by Sina et al. [15] and Mahi et al. [16] only a few years ago. The equation of deflection was derived applying Hamilton's principle. The Galerkin method was employed to analyse free vibration of sandwich beams with FG core in [17].

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Despite the variety of methods and approaches for analytical and computational analysis of non-uniform and FG beams, no simple and fast solutions applicable for both free and forced vibrations in such beams with different boundary conditions and varying cross-section area were proposed. Only few solutions are found for the studies on the axially FG beams. Hereof, the purpose of the present work is to introduce the Haar wavelet approach for calculating natural frequencies in non-uniform and FG beams. The paper is organized in five sections. Section 2 describes integration of Haar wavelets. In Section 3, the problem and the solution are stated. Various numerical examples can be found in Section 4. The main conclusions are drawn in Section 5.

2. Integration of Haar wavelets

The Haar wavelet is one of the simplest wavelets which is discontinuous and resembles a step function. In other words, the Haar wavelets belong to the special class of discrete orthonormal wavelets. The other wavelets generated from the same mother wavelet form a basis whose elements are orthonormal to each other and are normalized to unit length. This property allows each wavelet coefficient to be computed independently of other wavelets. The Haar wavelet family for $\xi \in [0, 1]$ is defined as follows:

$$h_i(\xi) = \begin{cases} 1 & \text{for } \xi \in [\xi^{(1)}, \xi^{(2)}], \\ -1 & \text{for } \xi \in [\xi^{(2)}, \xi^{(3)}], \\ 0 & \text{elsewhere.} \end{cases} \tag{1}$$

In (1), notations

$$\xi^{(1)} = \frac{k}{m}, \qquad \xi^{(2)} = \frac{k+0.5}{m}, \qquad \xi^{(3)} = \frac{k+1}{m} \tag{2}$$

are introduced. Integer $m = 2^j$ ($j = 0, 1, \ldots, J$) is the factor of scale; $k = 0, 1, \ldots, m-1$ is the factor of delay. Integer J determines the maximal level of resolution. Index i in (1) is calculated as $i = m + k + 1$; the minimal value for i is one (if $j = 0$, then $m = 1$, $k = 0$); the maximal value of i is $2M$, which is 2^{J+1}. If index i is equal to one, the corresponding scaling function is $h_1(\xi) = 1$ if $\xi \in [0, 1]$, and $h_1(\xi) = 0$ elsewhere.

In [18], the Haar coefficient matrix $H_{(2M\times 2M)}(i, l) = h_i(\xi_l)$ is introduced; the collocation points are defined as:

$$\xi_l = \frac{l-0.5}{2M}, \quad l = 1, 2, \ldots, 2M. \tag{3}$$

For further studies, the integrals of the wavelets

$$p_{\alpha,i}(\xi) = \int_0^{\xi} p_{\alpha_{i-1,i}(\xi)} \mathrm{d}\xi \tag{4}$$

are required. In (4), $p_{0,i}(\xi) = h_i(\xi)$. These integrals are calculated analytically [19]. In case $i = 1$, the integral of the wavelet is $p_{\alpha,1}(\xi) = \xi^{\alpha}/\alpha!$, and in case $i > 1$ is

$$p_{\alpha,i}(\xi) = \begin{cases} 0 \quad \text{for } \xi < \xi^{(1)}, \\ \dfrac{1}{\alpha!}\left(\xi - \dfrac{k}{m}\right)^{\alpha} \quad \text{for } \xi \in [\xi^{(1)}, \xi^{(2)}], \\ \dfrac{1}{\alpha!}\left[\left(\xi - \dfrac{k}{m}\right)^{\alpha} - 2(\xi - \xi^{(2)})^{\alpha}\right] \\ \quad \text{for } \xi \in [\xi^{(2)}, \xi^{(3)}], \\ \dfrac{1}{\alpha!}\left[\left(\xi - \dfrac{k}{m}\right)^{\alpha} - 2(\xi - \xi^{(2)})^{\alpha} + (\xi - \xi^{(3)})^{\alpha}\right] \\ \quad \text{for } \xi > \xi^{(3)}. \end{cases} \tag{5}$$

Values $p_{\alpha,i}(0)$ and $p_{\alpha,i}(1)$ should be calculated in order to satisfy the boundary conditions. Evaluating integrals (5) in the collocation points, the following form could be obtained

$$P^{(\alpha)}(i, l) = p_{\alpha,i}(\xi_l), \tag{6}$$

where $P^{(\alpha)}$ is a $2M \times 2M$ matrix. It should be noted that calculations of matrices $H(i, l)$ and $P^{(\alpha)}(i, l)$ must be carried out only once.

3. Problem statement and method of solution

Consider an axially graded Euler–Bernoulli beam with a variable cross-section of length L. In the present study, it is assumed that the material properties and cross-section of the beam vary continuously along the length. Introducing the quantities:

$$\xi = \frac{x}{L}, \qquad k^4 = \frac{\rho_0 A_0 \omega^2 L^4}{E_0 I_0}, \tag{7}$$

the equation of motion for transverse vibrations is given by

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left[E(\xi)I(\xi)\frac{\mathrm{d}^2 W(\xi)}{\mathrm{d}\xi^2}\right] - k^4 m(\xi)W(\xi) = 0, \quad \xi \in [0, 1], \tag{8}$$

where $W(\xi)$ is the transverse deflection, $m(\xi) = \rho(\xi)A(\xi)$ is the mass at position ξ, $E(\xi)I(\xi) = D(\xi)$ is the bending stiffness; $\rho(\xi)$ is the mass density of the beam material, $E(\xi)$ is the Young's modulus, $A(\xi)$ is the cross-section area and $I(\xi)$ is the moment of inertia at ξ. In (7), k is the dimensionless natural frequency, and ρ_0, A_0, E_0, I_0 denote the values of ρ, A, E, I at $\xi = 0$, respectively. In the present study, it is assumed that functions $E(\xi)$ and $I(\xi)$ have derivatives up to the second order. From (8), it yields that

$$\begin{aligned} &\frac{\mathrm{d}^4 W(\xi)}{\mathrm{d}\xi^4}E(\xi)I(\xi) + 2\frac{\mathrm{d}^3 W(\xi)}{\mathrm{d}\xi^3}\left[\frac{\mathrm{d}E(\xi)}{\mathrm{d}\xi}I(\xi) + \frac{\mathrm{d}I(\xi)}{\mathrm{d}\xi}E(\xi)\right] \\ &\quad + \frac{\mathrm{d}^2 W(\xi)}{\mathrm{d}\xi^2}\left[\frac{\mathrm{d}^2 E(\xi)}{\mathrm{d}\xi^2}I(\xi) + 2\frac{\mathrm{d}E(\xi)}{\mathrm{d}\xi}\frac{\mathrm{d}I(\xi)}{\mathrm{d}\xi} + \frac{\mathrm{d}^2 I(\xi)}{\mathrm{d}\xi^2}E(\xi)\right] \\ &\quad - k^4 W(\xi)\rho(\xi)A(\xi) = 0, \quad \xi \in [0, 1]. \end{aligned} \tag{9}$$

For a general case, the solution of (9) is not available. According to [18,19], a highest-order derivative is expanded into the Haar series instead of solving the differential equation. Therefore, it is assumed that the fourth derivative of the solution (9) is sought in the following form:

$$W^{IV}(\xi) = \sum_{i=1}^{2M} a_i h_i(\xi), \tag{10}$$

where a_i are unknown wavelet coefficients. Integrating (10) four times and taking into account (4) and (5), we obtain

$$\begin{aligned} W'''(\xi) &= \sum_{i=1}^{2M} a_i p_{1,i}(\xi) + W'''(0), \\ W''(\xi) &= \sum_{i=1}^{2M} a_i p_{2,i}(\xi) + W'''(0)\xi + W''(0), \\ W'(\xi) &= \sum_{i=1}^{2M} a_i p_{3,i}(\xi) + \frac{1}{2}W'''(0)\xi^2 + W''(0)\xi + W'(0), \\ W(\xi) &= \sum_{i=1}^{2M} a_i p_{4,i}(\xi) + \frac{1}{6}W'''(0)\xi^3 \\ &\quad + \frac{1}{2}W''(0)\xi^2 + W'(0)\xi + W(0). \end{aligned} \tag{11}$$

In (11), quantities $W(0), W'(0), W''(0), W'''(0)$ can be evaluated from the boundary conditions. In the present work, the following boundary conditions are considered:

(i) Cantilever beams (CF)

In this case, one end $\xi = 0$ is clamped, while the other end $\xi = 1$ is free. The boundary conditions for the beam are $W(0) =$