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Short communication

# An elementary derivation of basic equations of the Reissner and Mindlin plate theories

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ABSTRACT

The basic equations of the Reissner and Mindlin plate theories are derived in an elementary way from the underlying assumptions of the theories about the distribution of in-plane stresses in the Reissner case and the distribution of displacement components in the Mindlin case. The derived equations include a parameter which allows the interpretation of the theories as an approximation of isotropic plates or transversally inextensible plates. Qualitative comportment between the theories is also given.


## 1. Introduction

This short communication is motivated by the articles of Wang et al. [1] and Batista [2] wherein the authors emphasize the practical importance of the Reissner and Mindlin plate theories resulting from their simplicity and notice that despite the fact that the theories rely on different assumptions they are often treated as a single theory, referred to as the Reissner–Mindlin plate theory. They, among others, qualitatively describe the differences between the theories and state that one difference between them is that the Reissner plate theory was derived from the variational principle of complementary strain energy. This is true for Reissner's primary derivation [3–5]; however, in his later paper [6] he abandoned variational principles and showed how his equations may be derived directly from elasticity equations by assumptions that the transverse shear stresses are parabolic and that the sum of the in-plane normal stresses is linearly distributed over the plate thickness. How Reissner's plate equations may be derived directly from elasticity equations without using variational principles was, however, previously shown by Green ([7], 224–229), who used Reissner's weighted displacements [5] and assumed that transverse shear stress components are parabolically distributed across the plate thickness. Variants of such derivation may be found in the books of Timoshenko ([8], 168–171), Girkmann ([9], 583–591) and Panc ([10], 34–41). How Reissner's equation for an unloaded plate may be derived without any special assumptions was recently shown by the present author [2]. Nevertheless, we may find statements alluding to an intrinsic connection between Reissner's plate theory with variational principles even in contemporary books (e.g. [11]).

As opposed to Reissner, who considered plate equilibrium, Mindlin, who followed Uflyand [12], considered the vibration of plates [13], and he derived his plate equations directly from elasticity equations without using variational principles. The static case of equations was previously deduced by Bolle [14] and the case with variational principles was deduced by Hencky [15].

In what follows, the theories are visited one more time, showing how the governing equations of the theories may be derived in an elementary way. In the derivation of equations only the basic assumptions of theories are retained: so in this paper the Reissner plate theory means the theory for which the basic equations are derived by assumption that the in-plane stresses are linearly distributed across the plate thickness, while the Mindlin plate theory means the theory where the in-plane displacements are assumed to be linearly distributed across the plate thickness.

## 2. The shear deformable plate theories

*Governing elasticity equations*. In describing a plate, a rectangular Cartesian coordinate system with coordinates $(x, y, z)$ is used. The coordinate $z$ is perpendicular to the plane of the plate and the plate faces are at $z = \pm h/2$. For a plate in equilibrium the following stress equilibrium equations must be satisfied:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \quad (x \rightleftarrows y) \quad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \quad (1)$$

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where $\sigma_x$, $\sigma_y$, $\sigma_z$, are the normal stress components, and $\tau_{xy}$, $\tau_{xz}$, $\tau_{yz}$ are the shear stress components. Here and in what follows the symbol $x \rightleftarrows y$ on the right-hand side of an equation means that the equation for the other coordinate is obtained by interchanging $x$ and $y$. Since we want to consider an elastic isotropic plate as well as an elastic transversally inextensible plate, we write the constitutive equations which connect the stress components with the displacement components $u$, $v$, $w$ in the following half-inverted form:

$$\frac{\partial w}{\partial z} = \frac{\omega}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] \quad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\tau_{xz}}{G} \quad (x \rightleftarrows y) \tag{2}$$

$$\sigma_x = \frac{E}{1-\nu^2}\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right) + \frac{\nu\omega}{1-\nu}\sigma_z \quad (x \rightleftarrows y)$$
$$\tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{3}$$

where

$$\omega = \begin{cases} 0 & \text{transversally inextensible plate} \\ 1 & \text{isotropic plate} \end{cases} \tag{4}$$

and where $E$ is the modulus of elasticity, $\nu$ is the Poisson ratio, and $G \equiv E/2\,(1+\nu)$ is the shear modulus, assumed to be isotropic. It is worthwhile mentioning that the transversally inextensible plate was introduced by Kromm [16], but neither Reissner [3–5] or Mindlin [13] discuss this possibility in their derivations.

The boundary conditions on the plate faces are

$$\tau_{xz}\,(x, y, \pm h/2) = \tau_{yz}\,(x, y, \pm h/2) = 0$$
$$\sigma_z\,(x, y, \pm h/2) = \pm\frac{p}{2} \tag{5}$$

where $p \equiv \sigma_z\,(x, y, h/2) - \sigma_z\,(x, y, -h/2)$. The last of these conditions is asymmetric with respect to coordinate $z$ and defines the pure bending of the plate. In this case the transverse displacement $w$ should be a symmetric function of $z$ and, as follows from the governing equations, the stress components $\sigma_x$, $\sigma_y$, $\sigma_z$, $\tau_{xy}$ and the displacement components $u$, $v$ are asymmetric functions of $z$ and the stress components $\tau_{xz}$, $\tau_{yz}$ are symmetric functions of $z$. Consequently, the stress components $\sigma_x$, $\sigma_y$, $\sigma_z$, $\tau_{xy}$ and the displacement components $u$, $v$ vanish at the plate middle plane, $z = 0$.

On a high level the elastic state of the plate is described by the stress resultants [8]

$$M_x \equiv \int_{-h/2}^{h/2} \sigma_x z\,dz \quad (x \rightleftarrows y) \quad M_{xy} \equiv \int_{-h/2}^{h/2} \tau_{xy} z\,dz$$
$$Q_x \equiv \int_{-h/2}^{h/2} \tau_{xz}\,dz \quad (x \rightleftarrows y) \tag{6}$$

where $M_x$, $M_y$ are the bending moments, $M_{xy}$ is the twisting moment and $Q_{xz}$, $Q_{yz}$ are the transverse shear forces, all on one unit of length. By means of the definitions (6) and the face boundary conditions (5), by integration of the stress equilibrium equation (1) across the plate thickness, we obtain the well-known plate equilibrium equations [8]:

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} \quad (x \rightleftarrows y) \quad \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -p. \tag{7}$$

The task of a shear deformable plate theory is now to find an *approximate* solution of the elasticity equations (1)–(3) subject to the face boundary conditions (5) which includes enough free functions by which a prescribed plate's edge boundary conditions may be satisfied. Besides that, it is required that an approximate solution satisfy the plate equilibrium equations (7). In what follows the superscripts R and M will be used to denote the Reissner and Mindlin plate quantities, respectively. Also, the $z$ coordinate is normalized as follows:

$$\zeta \equiv \frac{z}{h/2} \in [-1, 1]\,. \tag{8}$$

*Note* 1. The requirement of the satisfaction of plate equilibrium equations (7) is abandoned in some plate theories where equations are derived from the principle of virtual work [17]

*Note* 2. For the case $p = 0$ the exact solution of the stated problem exists, which may be found by the method of symbolic integration [18] or by the method of successive approximation [2].

*Reissner's plate theory*. As stated in the introduction, the basic assumption of Reissner's primary theory of plates is that the in-plane stresses are linearly distributed across the plate thickness [3,4]. Thus, by using the definition of stress resultants (6), the in-plane stress components may be written in the following well-known form:

$$\sigma_x^R = \frac{6M_x^R}{h^2}\zeta \quad (x \rightleftarrows y) \quad \tau_{xy}^R = \frac{6M_{xy}^R}{h^2}\zeta\,. \tag{9}$$

By means of these expressions, by integration of the equilibrium equations (1) with respect to $z$, use of the equilibrium equation $(7)_2$and plate face boundary conditions (5), we obtain the transverse stress components:

$$\tau_{xz}^R = \frac{3Q_x^R}{2h}\left(1-\zeta^2\right) \quad (x \rightleftarrows y) \quad \sigma_z^R = \frac{p}{4}\zeta\left(3-\zeta^2\right). \tag{10}$$

The displacement components may now be obtained by the integration of constitutive equations (2) as is shown in [2]. However, Reissner at this point made another approximation:

$$w = w_0^R\,(x, y)\,. \tag{11}$$

By this, the constitutive equation $(2)_1$ may not generally be satisfied unless a plate is transversally inextensible. Now, following integration of the constitutive equation $(2)_2$ with respect to $z$, by means of $(10)_1$, (11) and condition $u\,(x, y, 0) = 0$ we find the expressions for the in-plane displacement components:

$$u^R = -\frac{h}{2}\frac{\partial w_0^R}{\partial x}\zeta + \frac{Q_x^R}{4G}\zeta\left(3-\zeta^2\right) \quad (x, u \rightleftarrows y, v)\,. \tag{12}$$

In this way all the stress components and in-plane displacements are expressed by six unknowns: $M_x^R, M_y^R, M_{xy}^R, Q_x^R, Q_y^R$ and $w_0^R$. For the determination of these unknowns we have at our disposal three plate equilibrium equations (7); hence three more equations must be deduced from the remaining three constitutive equations (3) in such a way that these constitutive equations are satisfied in some approximate way. *This may be achieved if we require that the stresses produced by displacement* (12) *produce the same moments as the stress components* (9). So, by substituting (12) into constitutive equation (3), equating the results with (9) and performing integration over the plate thickness the three missing equations are available:

$$\begin{aligned} M_x^R &= -D\left(\frac{\partial^2 w_0^R}{\partial x^2} + \nu\frac{\partial^2 w_0^R}{\partial y^2}\right) + \frac{h^2}{5}\frac{\partial Q_x^R}{\partial x} \\ &\quad - \frac{h^2}{10}\frac{\nu\,(2-\omega)}{1-\nu}p \quad (x \rightleftarrows y) \\ M_{xy}^R &= -(1-\nu)\,D\frac{\partial^2 w_0^R}{\partial x\partial y} + \frac{h^2}{10}\left(\frac{\partial Q_x^R}{\partial y} + \frac{\partial Q_y^R}{\partial x}\right) \end{aligned} \tag{13}$$

where $D = h^3E/12\left(1-\nu^2\right)$ is the plate bending stiffness. For $\omega = 1$ these equations become Reissner's equations for the moments [3,4].