Engineering Structures 32 (2010) 2440–2454

Contents lists available at ScienceDirect

Engineering Structures

journal homepage: www.elsevier.com/locate/engstruct

# Second-order slope–deflection equations for imperfect beam–column structures with semi-rigid connections

J. Darío Aristizabal-Ochoa *

*School of Mines, National University, A.A. 75267, Medellin, Colombia*

ARTICLE INFO

*Article history:*
Received 19 February 2010
Received in revised form
2 April 2010
Accepted 6 April 2010
Available online 7 May 2010

*Keywords:*
Beams
Beam–columns
Initial imperfections
Imperfect columns
Frames
Large deflections
Nonlinear analysis
Semi-rigid connections
Second-order analysis
Stability
Structural analysis

ABSTRACT

A new set of second-order slope–deflection equations for Euler–Bernoulli beam–columns of symmetric cross-section including the effects of initial imperfections (i.e., initial curvature, out-of-plumbness and axial load eccentricities) and semi-rigid connections is developed in a classical manner. The proposed method has the following advantages: (1) it can be utilized in the stability and second-order analysis of framed structures made of Euler–Bernoulli imperfect beam–columns with rigid, semi-rigid, and simple connections subject to axial and transverse loads; (2) the effects of semi-rigid connections and member imperfections are condensed into the slope–deflection equations for tension and compression axial loads; (3) it is more accurate than any other method available in the technical literature and capable of capturing not only the first-order and second-order elastic responses of frames made of imperfect beam–columns but also the phenomenon of reversals of deflections along the members as the axial loads are increased; and (4) it is powerful, practical, versatile and easy to apply. Analytical studies indicate that the initial imperfections (a) act as if they were additional transverse loads proportional to the bending stiffness and magnitudes of the imperfections of the corresponding beam–column; and (b) are detrimental to structures, increasing the lateral deflections, moments, and shears, and also reducing the critical axial loads of beam–columns and framed structures. This is particularly critical in structures made of beam–columns with initial crookedness and low stiffness connections subjected to high compressive axial loads. In addition, the effects of initial curvature are amplified by the compressive axial loads applied at the ends of the member. Four comprehensive examples are included that show the effectiveness of the proposed method.



## 1. Introduction

Current trends for lighter and slender structures have created great interest in the stability and second-order analyses of structures, particularly those with semi-rigid connections. These topics have been investigated by Aristizabal-Ochoa [1–4] using both the classical and the “modified” stability functions as defined by Timoshenko and Gere [5]. However, the stability of framed structures using the classical slope–deflection method including the effects of initial imperfections, *P*–Delta effects, and semi-rigid connections is not yet available in the technical literature.

The slope–deflection method (SDM) is a general method used in the analysis of statically indeterminate beam structures with rigid joints subjected to transverse loads causing bending and shear. The classical SDM equations were derived by Wilson and Maney in 1915 [6] by means of the moment-area theorems considering bending deformations and neglecting those due to shear and axial forces. Basically, a number of simultaneous equations are formed with the unknowns taken as the angular rotations and displacements of each joint. Once these equations have been solved, the bending moments at all joints may be determined. The SDM, which represents a turning point in the evolution and development of the matrix stiffness method as known today, is relatively simple to explain and apply since it is based on equilibrium of the joints and members [7]. The SDM is generally taught in preliminary structural analysis courses [8] and commonly used in structural design [9] because it provides a clear perspective and a complete understanding of how the internal moments and the corresponding deformations are interrelated, a concept that is essential in structural engineering.

The main objective of this paper is to present a new set of slope–deflection equations for the stability and second-order analysis of plane-framed structures made of imperfect beam–columns of symmetrical cross-section subjected to both transverse and axial loads, including the combined effects of eccentricities of the axial loads, initial curvature, out-of-plumbness, and semi-rigid connections. The proposed method which is based on the classic stability functions for Euler–Bernoulli beam–columns with semi-rigid connections [1,2] has the following advantages: (1) the effects of

* Tel.: +57 42686218; fax: +57 44255152.
*E-mail address:* jdaristi2@yahoo.com.

0141-0296/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.engstruct.2010.04.018

**Nomenclature**

- $E$ Young's modulus of the material.
- $h$ Length of the beam–column AB.
- $I$ Principal moment of inertia of the beam–column about its axis of bending.
- $M$ Applied bending moment or bending moment diagram.
- $M_a$ and $M_b$ Bending moments (clockwise +) at ends A and B, respectively.
- $P$ Applied axial load (+compression, −tension) or axial load diagram.
- $P_{cr}$ Critical axial load.
- $P_e = \pi^2 EI/h^2$ Euler load.
- $R_a$ and $R_b$ Stiffness indices of the flexural connection at A and B, respectively.
- $u(x)$ Lateral deflection of the beam–column center line.
- $V$ Shear force diagram.
- $\Delta_o$ Initial out-of-plumbness of end B with respect to end A.
- $\Delta$ Sway of end B with respect to end A.
- $\kappa_a$ and $\kappa_b$ Flexural stiffness of the end connections at A and B, respectively.
- $\rho_a$ and $\rho_b$ Fixity factors at A and B of column AB, respectively.
- $\psi(x) = \mathrm{d}u/\mathrm{d}x$ Beam slope due to applied bending (Fig. 1(c)).
- $\psi_{a'}$ and $\psi_{b'}$ Bending rotations of cross-sections at ends A′ and B′ with respect to the original shape of the beam column A′B′, respectively.
- $\phi = \sqrt{|Ph^2/EI|}$ Stability function in the plane of bending.
- $\theta_a$ and $\theta_b$ Rotations of ends A and B due to bending with respect to the vertical axis, respectively:

$$\left[\theta_a = \psi_{a'} + \frac{M_a}{\kappa_a} \text{ and } \theta_b = \psi_{b'} + \frac{M_b}{\kappa_b}\right].$$

semi-rigid connections are condensed into the slope–deflection equations for tension or compression axial loads; (2) it is more accurate than any other method available in the technical literature capable of capturing the second-order response of framed structures made of beam–columns with initial imperfections and semi-rigid connections as well as the phenomenon of reversals of deflections along the members (see [5], p. 34); and (3) the method is powerful, practical, versatile and easy to teach. The effects of axial and shear deformations along the member are assumed negligible. Four comprehensive examples are presented that show the effectiveness of the proposed method and the corresponding equations.

## 2. Structural model

### 2.1. Assumptions

Consider the two-dimensional (2D) prismatic beam–column shown in Fig. 1. The element is made up of the linear elastic Euler–Bernoulli beam–column itself, AB, and lumped semi-rigid connections at ends A and B as shown by Fig. 1(a) with bending stiffness $\kappa_a$ and $\kappa_b$, respectively. It is assumed that (1) the beam–column AB of height $h$ is made of a linear elastic material with elastic modulus $E$; (2) the centroidal axis of the beam–column is not a perfect straight line but has an initial crookedness defined by a parabola that is symmetric about its midspan, $u_1 = \frac{4a}{h^2}x(h-x)$, or by a series of sinusoidal waves, $u_1 = \sum_{n=1}^{n=\infty} a_n \sin\left(n\pi\frac{x}{h}\right)$, and it has an out-of-plumbness defined by the lateral sway $\Delta_o$, as shown in Fig. 1; (3) the beam–column AB is subjected simultaneously to transverse loads and axial load $P$ with eccentricities $e_a$ and $e_b$ at ends A and B, respectively. All applied forces and initial imperfections cause bending about the principal axis of the member; (4) the beam–column is prismatic with symmetric cross-section and principal moment of inertia $I$ in the plane of bending; and (5) the strains and deformations along its span are relatively small so that the principle of superposition can be applied.

The bending connections at both ends are elastic, with stiffnesses $\kappa_a$ and $\kappa_b$ (whose units are in force–distance/radian) in the plane of bending of the beam–column. The ratios $R_a = \kappa_a/(EI/h)$ and $R_b = \kappa_b/(EI/h)$ are denoted as the *stiffness indices* of the connections at ends A and B, respectively. These indices vary from zero (i.e., $R_a = R_b = 0$) for simple connections (i.e., pinned) to infinity (i.e., $R_a = R_b = \infty$) for fully restrained connections (i.e., rigid). Notice that the proposed algorithm can be utilized in the inelastic analysis of beam–columns when the inelastic behavior is concentrated at the end connections. This can be carried out by updating the stiffnesses $\kappa_a$ and $\kappa_b$ of the connections for each load increment in a linear-incremental fashion.

For convenience, the following two parameters are introduced:

$$\rho_a = \frac{1}{1+\frac{3}{R_a}}; \quad \text{and} \quad \rho_b = \frac{1}{1+\frac{3}{R_b}}. \tag{1a–b}$$

$\rho_a$ and $\rho_b$ are called the *fixity factors*. For hinged connections, both the fixity factor $\rho$ and the rigidity index $R$ are zero; but for rigid connections, the fixity factor is 1 and the rigidity index is infinity. Since the fixity factor can only vary from 0 to 1 for elastic connections (while the rigidity index $R$ may vary form 0 to $\infty$), it is more convenient to use in the elastic analysis of structures with semi-rigid connections.

### 2.2. Second-order analysis

The second-order lateral deflection ($u$) and corresponding slope ($\mathrm{d}u/\mathrm{d}x$) along the height of the column caused by the axial load $P$ (assuming that $e_a = e_b = \Delta = \Delta_o = 0$) with initial curvature given by a parabola or by a series of sinusoidal waves are derived below in Sections 2.2.1 and 2.2.2, respectively. Also below in Section 2.2.3, a complete second-order analysis is carried out on a partially braced column (Fig. 1(a)) with semi-rigid connections and initial out-of-plumbness $\Delta_o$ subjected to axial load $P$ with eccentricities $e_a$ and $e_b$ at ends A and B, respectively.

#### 2.2.1. Second-order deflection and slope assuming an initial imperfection of parabolic shape $u_1 = \frac{4a}{h^2}x(h-x)$

Knowing that

$$M = -EI\frac{\mathrm{d}^2u}{\mathrm{d}x^2},$$

then

$$-EI\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = P\left[u + \frac{4a}{h^2}x(h-x)\right]$$

or

$$EI\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + Pu = -\frac{4a}{h^2}x(h-x)P. \tag{2}$$

Applying the following boundary conditions: $u(0) = u(h) = 0$, then the solution to Eq. (2) is given by

$$u(x) = \frac{8a}{\phi^2}\left[\frac{\cos[(1-2x/h)\,\phi/2]}{\cos(\phi/2)} - 1\right] - \frac{4a}{h^2}x\,(h-x)\,. \tag{3}$$

Therefore,

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{8a}{\phi h}\left[\frac{\sin[(1-2x/h)\,\phi/2]}{\cos(\phi/2)}\right] - \frac{4a}{h^2}\,(h-2x) \tag{4}$$

where $\phi^2 = Ph^2/EI$ and $a$ is the initial camber at the midspan.