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In-plane thermoelastic behaviour and buckling of pin-ended and fixed circular arches

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ABSTRACT

This paper presents a systematic treatment of the in-plane thermoelastic analysis of pin-ended and fixed circular arches that are subjected to a uniform temperature field. A virtual work method is used to develop the differential equations of equilibrium. It is found that the membrane stresses and strains in arches that are subjected to a uniform temperature field form a special state for which compressive stresses are associated with tensile strains. It is also found that the thermal effects of the uniform temperature field on shallow arches are significant. The strains and stresses produced by the uniform temperature field in shallow arches are much higher that those in deep arches. Under a uniform temperature field, the central radial deflection of shallow arches may exceed the deflection limit for serviceability and the stresses in shallow arches may reach the yield stress of the material. The in-plane thermoelastic bifurcation buckling of arches under a uniform temperature field is also investigated using classical buckling theory. It is found that, in practice, thermoelastic bifurcation buckling is possible only for very shallow arches. For most arches, the critical temperature for in-plane thermoelastic bifurcation buckling is quite high, so the central radial deflection of the arches may exceed the deflection limit for serviceability and the stresses in the arches may reach the yield stress of the material before the critical temperature for the thermoelastic buckling is reached.

1. Introduction

The thermoelastic analysis of end-restrained straight members has been presented in a number of books and papers, and the in-plane thermoelastic behaviour of restrained straight members is well understood [1–6]. The thermal expansion is restrained in pin-ended and fixed arches, and this produces equal and opposite horizontal reaction forces at both ends of pin-ended and fixed arches. The thermal expansion also produces equal and opposite moment reactions at both ends of fixed arches. These end-restraining reactions will produce axial compressive and bending actions in the arch, which in turn produce compressive stresses in many cases. On the other hand, with a uniform increase of temperature, an end-restrained arch tends to displace upwards and the length of the arch correspondingly increases. As a result, tensile strains may be produced in the arch. This forms a special case for which the compressive stresses in the arch produced by a uniform temperature field may be associated with tensile strains. Accurate information about the thermal deformations, strains and stresses is much needed for comprehensive serviceability limit state, strength limit state, and stability limit state designs of arches under a uniform temperature field. However, very few studies that consider the thermoelastic analysis of arches under a uniform temperature field appear to have been reported. Flexibility methods can be used for the in-plane thermoelastic analysis of arches, but they usually do not give the thermal deformations directly and they are difficult to treat systematically.

Although the axial compressive force produced by the uniform heating varies along the arch axis, an assumption that the axial compressive force in the arch produced by the uniform heating is uniform along the arch axis is sometimes used in the in-plane thermoelastic analysis for shallow arches [7,8]. This assumption greatly simplifies the thermoelastic analysis of arches, but it needs to be justified and the accuracy of the results based on this assumption need to be investigated more generally.

The primary radial displacements of a symmetric arch due to uniform heating are symmetrical. As the temperature increases, the symmetric radial displacements continue to increase, and at a certain critical temperature the symmetric radial displacements of an arch may bifurcate into antisymmetric displacements which lead to antisymmetric buckling of the arch. Bradford [7,8] used a nonlinear analytical method to investigate the nonlinear buckling of an axial-elastically restrained shallow arch under a uniform temperature field and found that nonlinear thermoelastic buckling of the shallow arch cannot occur under a uniform temperature field.

The aims of this paper are to present a systematic treatment of the in-plane thermoelastic analysis of pin-ended and fixed circular

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arches that are subjected to a uniform temperature field, and to investigate the thermoelastic in-plane bifurcation buckling of shallow arches using a classical buckling theory.

2. Basic assumptions

To facilitate the thermoelastic analysis, the following assumptions are used in this investigation:

1. The deformations of the arch are elastic and satisfy the Euler–Bernoulli hypothesis, i.e. the cross-section remains planar and perpendicular to the arch axis during deformation.
2. The states of deformation and temperature are treated as time independent, i.e. the derivative of the temperature T with respect to the time t vanishes: $\mathrm{d}T/\mathrm{d}t = 0$; and so this separates the analysis of the temperature field from that of the displacement field and makes the problem uncoupled.
3. The temperature field $\Delta T = (T - 20\,^\circ\mathrm{C})$ is uniform over the entire arch, i.e. the temperature gradient also vanishes: $\nabla(\Delta T) = 0$, where T is the uniform temperature and the ambient temperature is assumed to be 20 °C.
4. The coefficient of thermal expansion α is independent of the temperature T.
5. The arches are assumed to be slender, i.e. the dimensions of the cross-section are much smaller than the length and radius of the arch.
6. Because the thermoelastic analysis of slender arches is carried out with the same degree of rigour as that accepted in the theory of elasticity, the expansions of the cross-sections are assumed to be so small that they can be disregarded in the analysis [1,2,4,6].
7. The arches are assumed to be fully restrained in the lateral direction.

3. Thermoelastic analysis

3.1. Differential equations of equilibrium

An in-plane linear analysis is developed to investigate the thermoelastic behaviour of a circular arch that is subjected to a uniform temperature field ΔT. It is known that due to the coupling between the axial and radial deformations, the strain at an arbitrary point of the cross-section of the arch includes the coupling terms and can be expressed as the sum of membrane strain ϵ_m and bending strain ϵ_b as [9–11]

$$\epsilon = \epsilon_m + \epsilon_b \quad \text{with}\ \epsilon_m = \tilde{w}' - \tilde{v} \quad \text{and} \quad \epsilon_b = -\frac{y}{R}(\tilde{v}'' + \tilde{w}'), \tag{1}$$

where $(\,)' = \mathrm{d}(\,)/\mathrm{d}\theta$, $(\,)'' = \mathrm{d}^2(\,)/\mathrm{d}\theta^2$, θ is the angular coordinate (Fig. 1), the dimensionless displacements $\tilde{v}$ and $\tilde{w}$ are defined by $\tilde{v} = v/R$ and $\tilde{w} = w/R$, v and w are the radial and axial displacements in the directions of axes oy and os, respectively, y is the coordinate of the point in the axis oy, and R is the radius of the arch. The positive direction of the axis oy is toward the centre of the circular arch, as shown in Fig. 1, and so its direction changes around the arch axis os and is always perpendicular to the tangent of the axis os.

From the Duhamel–Neumann equation [4], the stress produced by the uniform temperature field ΔT can be expressed as

$$\sigma = E(\epsilon - \alpha\Delta T) \tag{2}$$

in which E is the Young's modulus at the temperature T and A is the area of the cross-section.

The differential equations of equilibrium for the thermoelastic analysis of an arch under a uniform temperature field ΔT can be

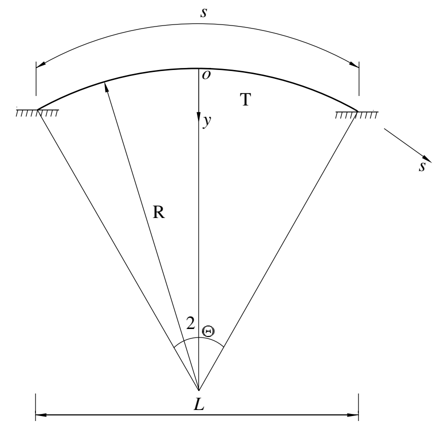

Fig. 1. Shallow arch under uniform radial loading and uniform temperature.

derived by using the principle of virtual work, which can be stated as

$$\delta\Pi = \int_V \sigma\,\delta\epsilon\,\mathrm{d}V = 0, \tag{3}$$

for all arbitrary variations of the admissible deformations, where V indicates the volume of the arch, the strain ϵ and stress σ are given by Eqs. (1) and (2), and $\delta(\,)$ denotes the Lagrange operator of simultaneous variations.

By substituting Eqs. (1) and (2), the statement of the principle of virtual work given by Eq. (3) can be written as

$$\int_{-\Theta}^{\Theta} [NR(\delta\tilde{w}' - \delta\tilde{v}) - M(\delta\tilde{v}'' + \delta\tilde{w}')]\mathrm{d}\theta = 0, \tag{4}$$

where the axial force N at a cross-section is given by

$$N = \int_A \sigma\,\mathrm{d}A = AE\left(\tilde{w}' - \tilde{v} - \alpha\Delta T\right), \tag{5}$$

and the bending moment M at a cross-section is given by

$$M = \int_A \sigma y\,\mathrm{d}A = -\frac{EI_x}{R}\left(\tilde{v}'' + \tilde{w}'\right) \tag{6}$$

in which I_x is the second moment of area of the cross-section about its major principal axis.

Integrating Eq. (4) by parts leads to the differential equations of equilibrium as

$$M'' + NR = 0 \tag{7}$$

for the radial direction, and

$$M' - (NR)' = 0 \tag{8}$$

for the axial direction; and to the static boundary conditions for pin-ended arches as

$$M = 0 \quad \text{at}\ \theta = \pm\Theta. \tag{9}$$

By substituting Eqs. (5) and (6), the differential equations of equilibrium Eqs. (7) and (8) become

$$-EI_x\left(\tilde{v}^{iv} + \tilde{w}'''\right) + R^2 AE\left(\tilde{w}' - \tilde{v} - \alpha\Delta T\right) = 0, \tag{10}$$

and

$$-EI_x(\tilde{v}''' + \tilde{w}'') - R^2 AE\left(\tilde{w}'' - \tilde{v}'\right) = 0; \tag{11}$$

and the static boundary condition given by Eq. (9) becomes

$$-EI_x\left(\tilde{v}'' + \tilde{w}'\right) = 0 \quad \text{at}\ \theta = \pm\Theta. \tag{12}$$