Engineering Structures 30 (2008) 3355–3364

Contents lists available at ScienceDirect

Engineering Structures

journal homepage: www.elsevier.com/locate/engstruct

Structural analysis of a curved beam element defined in global coordinates

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ARTICLE INFO

Article history:
Received 18 February 2008
Received in revised form
24 April 2008
Accepted 16 May 2008
Available online 25 June 2008

Keywords:
Curved beam
Global coordinates
Structural analysis
Frenet frame
Differential system
Stiffness matrix
Transfer matrix
Arches

ABSTRACT

In this article, a system of twelve differential equations expressed in the global Cartesian coordinate system to simulate the structural behavior of a general curved beam element, is presented. Different shape geometry of the curved centroid line, shearing deformations, varying cross section area, non-symmetric section and generalized loads are taken into account. The lower-triangular form of the system of equations permits the determination of analytical results through successive simple integrations row by row. Exact analytical solutions and expressions of transfer and stiffness matrices for widely spread cases of curved beams such as the circular arch and balcony, are provided. Likewise, numerical accurate results for the case of variable cross-section cantilever and circular helical beam are given in the examples for verification.

1. Introduction

There exists much literature on structural analysis of curved beam elements [1–6]. Most of the authors approach this problem of twisted elements, expressing the functions in natural coordinates using the Frenet frame system of reference [7,8]. These models to simulate the mechanical behavior of the problem, could be in terms of virtual works and energy methods [9–11]; in separate equations of equilibrium and kinematics [12–15]; or in terms of a system of equations [16,17]. Particularly, the circular arch is a problem very spread in this field [18–21]. Other types of curved arch element as parabola and helix have been considered [22–26]. We explicitly mention Yu and his group's work, for given systematic research on curved beams, studying tangential and shearing stresses in two-material curved beams [27–29] and the generalized coordinate for warping of naturally curved and twisted beams with general cross-sectional shapes [30].

In the state of the art, the independent variable for curved beams has been the arc length (naturally equations) or another parameter (non-naturally equations), but always using the Frenet mobile frame.

The authors that subscribe this article, presented a general formulation for naturally [31] and non-naturally [32] curved beam elements, taking into account shearing deformations, varying cross section area, non-symmetric section and generalized loads.

None of the above formulations have considered the curved element in global Cartesian coordinates. The objective of this article is to present this new system of differential equations and show the capabilities and advantages respect with the other models. The lower-triangular form of the system of equations permits the determination of analytical results through successive simple integrations row by row. Exact analytical solution and expressions of transfer and stiffness matrices for well-known cases of curved beams such as the circular arch and balcony, are provided. Variable cross-section cantilever and a circular helical beam are also compared with results given in the literature for verification purposes.

2. Formulation. Differential systems

A curved beam is generated by a plane cross-section which centroid P sweeps perpendicularly through all the points of an axis line. The vector radius $\mathbf{r} = \mathbf{r}(s)$ expresses this curved line, where s (m) length of the arc, is the independent variable of the structural problem. The reference coordinate system used to represent the intervening known and unknown functions of the problem is the Frenet frame P_{tnb} [33]. Its unit vectors tangent $\mathbf{t}$, normal $\mathbf{n}$ and binormal $\mathbf{b}$ are:

$$\mathbf{t} = D\mathbf{r} \qquad \mathbf{n} = \frac{D^2\mathbf{r}}{|D^2\mathbf{r}|} \qquad \mathbf{b} = \mathbf{t} \times \mathbf{n}$$

where, $D = \mathrm{d}/\mathrm{d}s$ is the derivative respect the parameter s.

The Frenet–Serret equations [34] describe the movement of the frame system along the axis line. They are obtained with the

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doi:10.1016/j.engstruct.2008.05.011

versors tangent, normal and binormal derivates with respect to the arc length. Its matricial expression is:

$$D\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix}=\begin{bmatrix}0 & \chi(s) & 0\\ -\chi(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix} \tag{1}$$

where $\chi = \chi(s)$ (m^{-1}) and $\tau = \tau(s)$ (m^{-1}) are the flexure and torsion curvatures respectively, which represent the natural equations of the centroid line.

Assuming the habitual principles and hypotheses of the strength of materials [35] and considering the stresses associated with the normal cross-section (σ, τ_n, τ_b) ($\mathrm{N/m^2}$), the geometric characteristics of the section are: area $A(s)$ ($\mathrm{m^2}$), shearing coefficients $\alpha_n(s), \alpha_{nb}(s), \alpha_{bn}(s), \alpha_b(s)$ and moments of inertia $I_t(s)$, $I_n(s)$, $I_b(s)$, $I_{nb}(s)$ ($\mathrm{m^4}$). Longitudinal $E(s)$ ($\mathrm{N/m^2}$) and transversal $G(s)$ ($\mathrm{N/m^2}$) elasticity moduli give the elastic condition of the material.

Applying the equilibrium of forces, the following equation is obtained:

$$\begin{bmatrix}D & -\chi & 0\\ \chi & D & -\tau\\ 0 & \tau & D\end{bmatrix}\begin{bmatrix}N\\ V_n\\ V_b\end{bmatrix}+\begin{bmatrix}q_t\\ q_n\\ q_b\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}. \tag{2}$$

The vectors involved in the equilibrium (Fig. 1a) are

Internal forces $\mathbf{V_t} = N\mathbf{t} + V_n\mathbf{n} + V_b\mathbf{b} = \int_A \sigma \mathrm{d}A\mathbf{t} + \int_A \tau_n \mathrm{d}A\mathbf{n} + \int_A \tau_b \mathrm{d}A\mathbf{b}$ (N).

Force load $\mathbf{q_t} = q_t\mathbf{t} + q_n\mathbf{n} + q_b\mathbf{b}$ (N/m).

The equation of moments is obtained applying the equilibrium law as well:

$$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}N\\ V_n\\ V_b\end{bmatrix}+\begin{bmatrix}D & -\chi & 0\\ \chi & D & -\tau\\ 0 & \tau & D\end{bmatrix}\begin{bmatrix}T\\ M_n\\ M_b\end{bmatrix}+\begin{bmatrix}m_t\\ m_n\\ m_b\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}. \tag{3}$$

In this case, the vectors are:

Internal moments $\mathbf{M_t} = T\mathbf{t} + M_n\mathbf{n} + M_b\mathbf{b} = \int_A (\tau_b n - \tau_n b)\,\mathrm{d}A\mathbf{t} + \int_A \sigma b \mathrm{d}A\mathbf{n} - \int_A \sigma n \mathrm{d}A\mathbf{b}$ (Nm).

Moment load $\mathbf{m_t} = m_t\mathbf{t} + m_n\mathbf{n} + m_b\mathbf{b}$ (Nm/m).

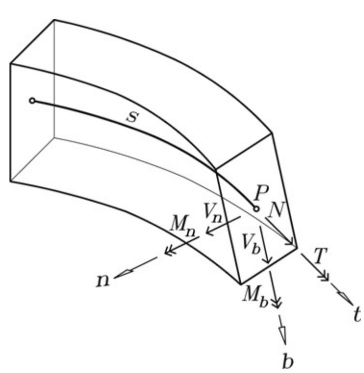

Fig. 1a. Internal forces and moments in Frenet frame.

Once the constitutive relations are defined, kinematics law relates the rotations and displacements (Fig. 1b):

$$\begin{bmatrix}-\dfrac{1}{GI_t} & 0 & 0\\ 0 & -\dfrac{I_b}{E(I_nI_b - I_{nb}^2)} & -\dfrac{I_{nb}}{E(I_nI_b - I_{nb}^2)}\\ 0 & -\dfrac{I_{nb}}{E(I_nI_b - I_{nb}^2)} & -\dfrac{I_n}{E(I_nI_b - I_{nb}^2)}\end{bmatrix}\begin{bmatrix}T\\ M_n\\ M_b\end{bmatrix}$$

$$+\begin{bmatrix}D & -\chi & 0\\ \chi & D & -\tau\\ 0 & \tau & D\end{bmatrix}\begin{bmatrix}\theta_t\\ \theta_n\\ \theta_b\end{bmatrix}-\begin{bmatrix}\Theta_t\\ \Theta_n\\ \Theta_b\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}. \tag{4}$$

Rotations components are given by $\boldsymbol{\theta}_\mathbf{t} = \theta_t\mathbf{t} + \theta_n\mathbf{n} + \theta_b\mathbf{b}$ (rad).

Rotation load $\boldsymbol{\Theta}_\mathbf{t} = \Theta_t\mathbf{t} + \Theta_n\mathbf{n} + \Theta_b\mathbf{b}$ (rad/m).

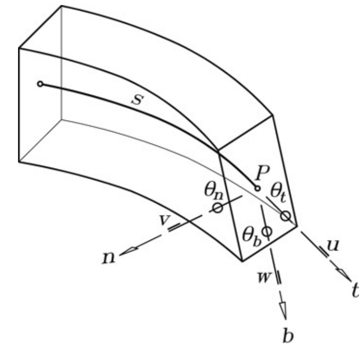

Fig. 1b. Deflections in Frenet frame.

Following the same procedure, the displacement equation is expressed:

$$\begin{bmatrix}-\dfrac{1}{EA} & 0 & 0\\ 0 & -\dfrac{\alpha_n}{GA} & -\dfrac{\alpha_{nb}}{GA}\\ 0 & -\dfrac{\alpha_{nb}}{GA} & -\dfrac{\alpha_b}{GA}\end{bmatrix}\begin{bmatrix}N\\ V_n\\ V_b\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\theta_t\\ \theta_n\\ \theta_b\end{bmatrix}$$

$$+\begin{bmatrix}D & -\chi & 0\\ \chi & D & -\tau\\ 0 & \tau & D\end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix}-\begin{bmatrix}\Delta_t\\ \Delta_n\\ \Delta_b\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \tag{5}$$

where displacement components are denoted as $\boldsymbol{\delta}_\mathbf{t} = u\mathbf{t} + v\mathbf{n} + w\mathbf{b}$ (m) and displacement load $\boldsymbol{\Delta}_\mathbf{t} = \Delta_t\mathbf{t} + \Delta_n\mathbf{n} + \Delta_b\mathbf{b}$ (m/m)

Eqs. (2)–(5) are related and they build the system of linear ordinary differential equations which simulates the structural behaviour of a curved beam element [31]: See Box I.

It is important to note the strict order of the twelve functions in the equation. Forces produce moments, moments produce rotations and rotations produce displacements, in terms of the load applied. All functions are interconnected. This arranged format has permitted to obtain directly numerical results and matrices expressions [36].

The system (6) given in Box I is associated to the Frenet frame in natural coordinates of the curved line.

It is possible to implement a change of basis and express the functions (Fig. 2) in a global coordinate system P_{xyz} which unit vectors are $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$:

$$\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix}=\begin{bmatrix}\upsilon_{tx} & \upsilon_{ty} & \upsilon_{tz}\\ \upsilon_{nx} & \upsilon_{ny} & \upsilon_{nz}\\ \upsilon_{bx} & \upsilon_{by} & \upsilon_{bz}\end{bmatrix}\begin{bmatrix}\mathbf{i}\\ \mathbf{j}\\ \mathbf{k}\end{bmatrix}.$$

The different coefficients of the basis change matrix represent the direction cosines.

The differential system (6) given in Box I is transformed into global Cartesian coordinates in Box II.

The components of internal forces, moments, rotations and displacements involved in Eq. (7) given in Box II are referred to the global absolute coordinate system.

This new general expression of the differential system, which simulates the structural behaviour of the linear element, has a lower-triangular form. That important property permits to solve analytically the differential equation system using successive integrations.

The analytic solution will exist if primitive functions of those integral equations are known. Different numerical methods for solving integral equations (simple numerical integration) are also suitable to reach accurate results [37].

Eq. (7) given in Box II can be particularized for curved beams in a plane. Assuming that the curve is contained in the xy plane with height $z = 0$, the unit vectors of the basis $\mathbf{k}$ and $\mathbf{b}$ are parallel and